Lecture 3

Solving Nonlinear Equations

Roots of the equation, Convergence

Ceng375 *Numerical Computations* at October 14, 2010

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Contents

Main Topics I

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1. **Interval Halving (Bisection)**. Describes a method that is very simple and foolproof but is not very efficient. We examine how the error decreases as the method continues.
2. **Linear Interpolation Methods**. Tells how approximating the function in the vicinity of the root with a straight line can find a root more efficiently. It has a better "rate of convergence".

Main Topics II

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4. **Muller's Method**. Approximates the function with a quadratic polynomial that fits to the function better than a straight line. This significantly improves the rate of convergence over linear interpolation.
5. **Fixed-Point Iteration:** $x = g(x)$ **Method**. Uses a different approach: The function $f(x)$ is rearranged to an equivalent form, $x = g(x)$. A starting value, x_0, is substituted into $g(x)$ to give a new x-value, x_1. This in turn is used to get another x-value. If the function $g(x)$ is properly chosen, the successive values converge.

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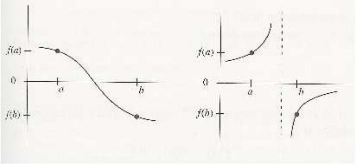

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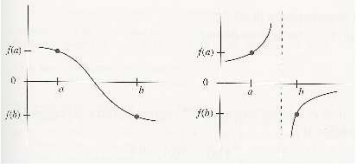

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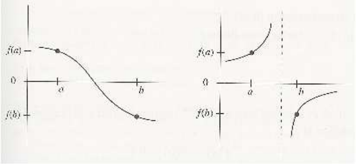

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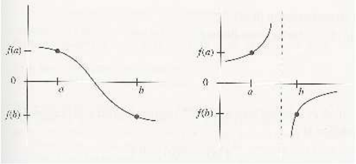

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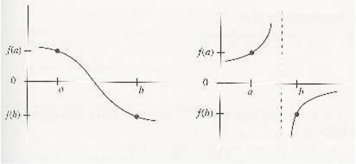

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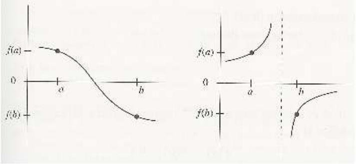

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- A plot of $f(x)$ is useful to know where to start.

Bisection II

An algorithm for halving the interval (Bisection):

To determine a root of $f(x) = 0$ that is accurate within a specified tolerance value, given values x_1 and x_2, such that $f(x_1) * f(x_2) < 0$,
Repeat
Set $x3 = (x_1 + x_2)/2$
If $f(x_3) * f(x_1) < 0$ Then
Set $x_2 = x_3$
Else Set $x_l = x_3$ End If
Until $(|x_1 - x_2|) < 2 * \textit{tolerance value}$

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- Think about the multiplication factor, 2.
- The final value of x_3 approximates the root, and it is in error by not more than $|x_l - x_2|/2$.
- The method may produce a false root if $f(x)$ is discontinuous on $[x_1, x_2]$.

Bisection III

```
>> format long e
>> fa=1e-120;fb=-2e-300;
>> fa*fb
ans =      0
>> sign(fa)~=sign(fb)
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- **Example**: Apply Bisection to $x - x^{1/3} - 2 = 0$.
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- Now, try with a user (you!) defined function;

$$f(x) = x - x^{1/3} - 2$$

```
>> brackPlot('fx3',?,?)
```

In both example, try with different intervals.

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>> f = inline ( ' 3 *x + sin ( x) - exp ( x) ')
>> fplot ( f, [ 0 2 ]) ; grid on
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Figure: Plot of the function: $f(x) = 3x + sin(x) - e^x$

- We see from the figure that indicates there are <u>zeros</u> at about $x = 0.35$ and 1.9.

Bisection VI

Table: The bisection method for $f(x) = 3x + sin(x) - e^x$, starting from $x_1 = 0, x_2 = 1$, using a tolerance value of 1E-4.

Iteration	X_1	X_2	X_3	$F(X_3)$	Maximum error	Actual error
1	0.00000	1.00000	0.50000	0.33070	0.50000	0.13958
2	0.00000	0.50000	0.25000	−0.28662	0.25000	−0.11042
3	0.25000	0.50000	0.37500	0.03628	0.12500	0.01458
4	0.25000	0.37500	0.31250	−0.12190	0.06250	−0.04792
5	0.31250	0.37500	0.34375	−0.04196	0.03125	−0.01667
6	0.34375	0.37500	0.35938	−0.00262	0.01563	−0.00105
7	0.35938	0.37500	0.36719	0.01689	0.00781	0.00677
8	0.35938	0.36719	0.36328	0.00715	0.00391	0.00286
9	0.35938	0.36328	0.36133	0.00227	0.00195	0.00091
10	0.35938	0.36133	0.36035	−0.00018	0.00098	−0.00007
11	0.36035	0.36133	0.36084	0.00105	0.00049	0.00042
12	0.36035	0.36084	0.36060	0.00044	0.00024	0.00017
13	0.36035	0.36060	0.36047	0.00013	0.00012	0.00005

- To obtain the true value for the root, which is needed to compute the actual error $\Rightarrow$ MATLAB

```
>> solve('3*x + sin(x) - exp(x)')
ans=
.36042170296032440136932951583028
```

Bisection VII

- A general implementation of bisection (**m-file: bisect.m**)

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- It is shown above how *brackPlot* can be combined with *bisect* to find a single root of an equation.
- The same procedure can be extended to find more than one root if more than root exists. Consider the code

```
xmin=...; xmax=...;
Xb=brackPlot('myFunction',xmin,xmax);
for k=1:size(Xb,1)
    x(k)=bisect('myFunction',Xb(k,:));
    fprintf('Suspected root at %f gives f(x)=%f \n'),
             x(k),myFunction(x(k)));
end
```

Use an appropriate 'myFunction', a suggestion is *sine* function.

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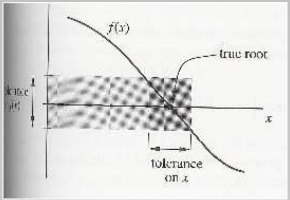

Figure: The stopping criterion for a root-finding procedure should involve a tolerance on x, as well as a tolerance on $f(x)$.

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- The major objection of interval halving has been that it is **slow to converge**.
- Bisection is generally recommended for finding an approximate value for the root, and then this value is refined by more efficient methods.

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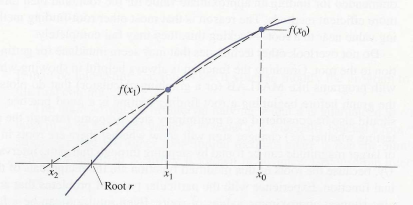

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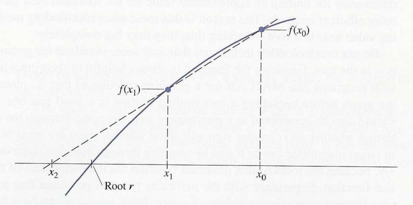

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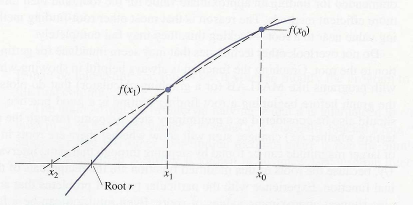

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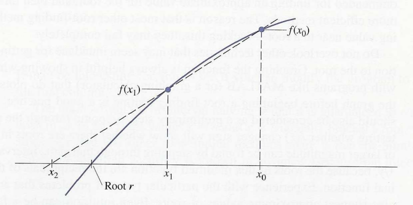

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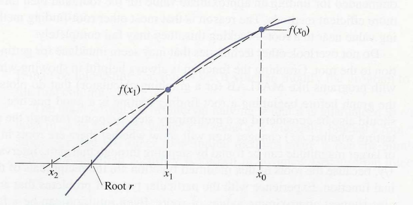

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- If $f(x)$ were truly linear, the straight line would intersect the x-axis at the root.

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- The net effect of this rule is to set $x_0 = x_1$ and $x_1 = x_2$, after each iteration.

Linear Interpolation Methods - The Secant Method III

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- **An algorithm for the Secant Method:**

To determine a root of $f(x) = 0$, given two values, x_0 and x_1, that are near the root,
If$|f(x_0)| < |f(x_1)|$ Then
Swap x_0 with x_1
Repeat
Set $x_2 = x_1 - f(x_1) * \frac{(x_0 - x_1)}{f(x_0) - f(x_1)}$
Set $x_0 = x_1$, Set $x_1 = x_2$
Until $|f(x_2)| <$ *tolerance value*

Linear Interpolation Methods - The Secant Method IV

Table: The Secant method for $f(x) = 3x + sin(x) - e^x$, starting from $x_0 = 1, x_1 = 0$, using a tolerance value of 1E-6.

Iteration	x_0	x_1	x_2	$f(x_2)$
1	1	0	0.4709896	0.2651588
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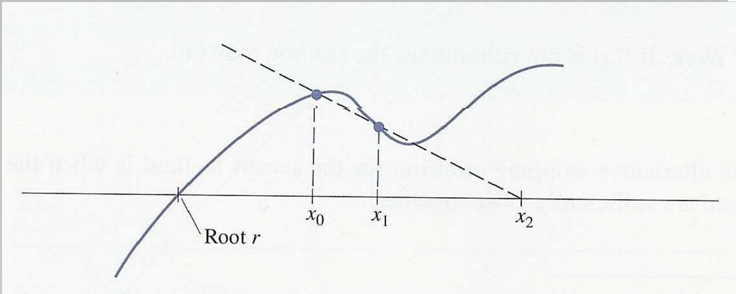
- If the function is far from linear near the root, the successive iterates can fly off to points far from the root, as seen if Fig. 5.



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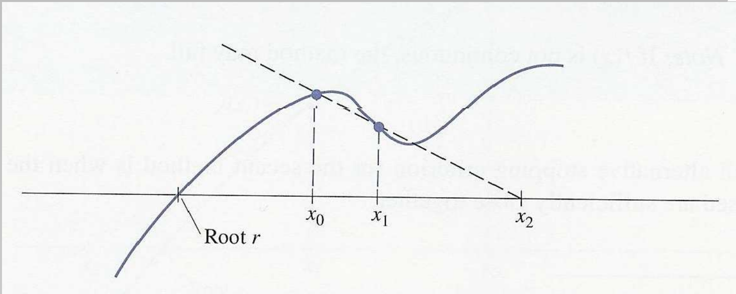

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- If the function was plotted before starting the method, it is unlikely that the problem will be encountered, because a better starting value would be used.

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- **An algorithm for the method of false position:**

To determine a root of $f(x) = 0$, given two values of x_0 and x_1 that bracket a root: that is, $f(x_0)$ and $f(x_1)$ are of opposite sign,
Repeat
Set $x_2 = x_1 - f(x_1) * \frac{(x_0 - x_1)}{f(x_0) - f(x_1)}$
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Set $x_1 = x_2$,
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Linear Interpolation Methods - False Position II

Table: Comparison of methods, $f(x) = 3x + sin(x) - e^x$, starting from $x_0 = 0, x_1 = 1$.

	Interval halving		False position		Secant method	
Iteration	x	$f(x)$	x	$f(x)$	x	$f(x)$
1	0.5	0.330704	0.470990	0.265160	0.470990	0.265160
2	0.25	−0.286621	0.372277	0.029533	0.372277	0.029533
3	0.375	0.036281	0.361598	$2.94 * 10^{-3}$	0.359904	$-1.29 * 10^{-3}$
4	0.3125	−0.121899	0.360538	$2.90 * 10^{-4}$	0.360424	$5.55 * 10^{-6}$
5	0.34375	−0.041956	0.360433	$2.93 * 10^{-5}$	0.360422	$3.55 * 10^{-7}$
Error after 5 iterations		0.01667		$-1.17 * 10^{-5}$		$< -1 * 10^{-7}$

(Exact value of root is 0.360421703.)

- Table 3 compares the results of three methods-interval halving (bisection), linear interpolation, and the secant method for $f(x) = 3x + sin(x) - e^x = 0$

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- Observe that the **speed of convergence** is best for the secant method, poorest for interval halving, and intermediate for false position.

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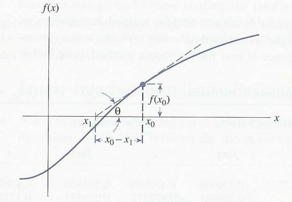

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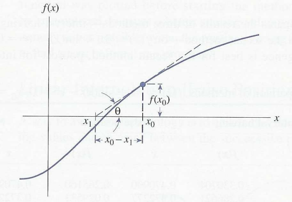

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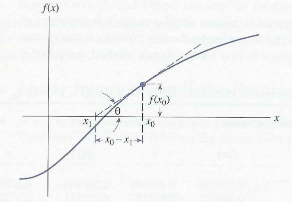

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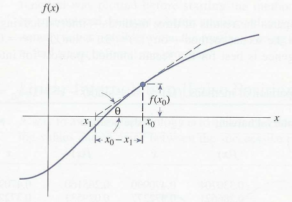

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- Like the previous ones, this method is also based on a <u>linear approximation</u> of the function, but does so using a <u>tangent to the curve</u> (see Figure 6).

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- This is continued until either the successive x-values are sufficiently close or the value of the function is sufficiently near zero.
- The calculation scheme follows immediately from the right triangle shown in Fig. 6.

$$tan\theta = f^{'}(x_0) = \frac{f(x_0)}{x_0 - x_1} \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

and the general term is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 0, 1, 2, \ldots$$