Lecture 5

Solving Sets of Equations I

Elimination Methods

Ceng375 *Numerical Computations* at November 4, 2010

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Contents

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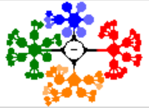

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1. **Matrices and Vectors.** Reviews concepts of matrices and vectors in preparation for their use.
2. **Elimination Methods.** Describes two classical methods that change a system of equations to forms that allow getting the solution by back-substitution and shows how the errors of the solution can be minimized.

Solving Sets of Equations II

3 **The Inverse of a Matrix and Matrix Pathology.** Shows how an important derivative of a matrix, its inverse, can be computed. It shows when a matrix cannot be inverted and tells of situations where no unique solution exists to a system of equations.

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5 **Iterative Methods.** It is described how a linear system can be solved in an entirely different way, by beginning with an initial estimate of the solution and performing computations that eventually arrive at the correct solution. An iterative method is particularly important in solving systems that have few nonzero coefficients.

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- When the term **vector** is used, it nearly always means a **column** vector.

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- Much more simply in matrix notation, as $Ax = b$ where

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- Example,

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 0 \\ -1 & 3 & 2 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \\ 2 \end{bmatrix}$$

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- is the same as the set of equations

$$\begin{aligned} &3x_1 + 2x_2 + 4x_3 = 14 \\ &x_1 - 2x_2 = -7 \\ &-x_1 + 3x_2 + 2x_3 = 2 \end{aligned}$$

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- An important property of an identity matrix, I, is that for any $n \times n$ (square) matrix, A,

$$I * A = A * I = A$$

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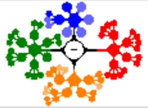

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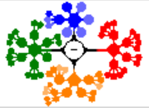

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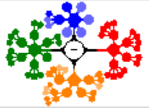

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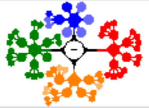

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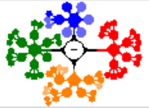
- Such a matrix is termed a **sparse matrix** and procedures that take advantage of this sparseness are of value.



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- Eigenvalue is a German word, the corresponding English term is characteristic value, but it is less frequently used.

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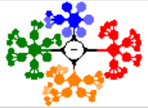

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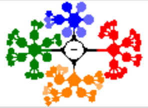

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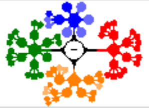

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Using Computer Algebra Systems

MATLAB can do matrix operations . We first define two matrices and a vector, *A, B,* and ν:

```
>> A = [ 4 1 -2 ; 5 1 3 ; 4 0 -1]
>> B = [ 3 3 1 ; -2 1 5 ; 2 2 0]
>> v = [ -2 3 1]
>> vt = v'
>> det(A)
>> trace (A)
We can get the characteristic polynomial;
>> poly(A)
ans =
1.000   -4.0000   2.0000   -21.0000
where the coefficients are given. This represents
x^3 - 4x^2 + 2x -21
```

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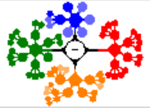
- To solve a set of linear equations.



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Dr. Cem Özdoğan

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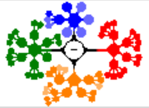

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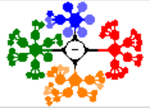

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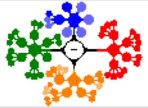

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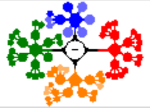

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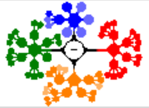

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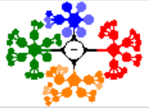

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- It also means that the determinant is zero. There is no solution.

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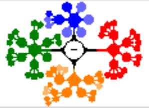

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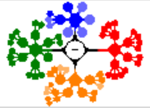

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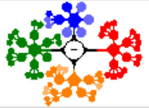

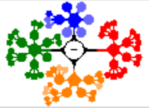

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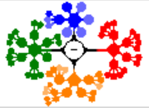

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- Back-substitution, gives us $x_3 = 3,\ x_2 = -2,\ x_1 = 2$

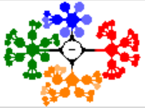
Gaussian Elimination III

Example m-file: Show steps in Gauss elimination and back substitution. No pivoting. (GEshow.m)

```
>> A=[4 -2 1 ; -3 -1 4 ; 1 -1 3];
>> b=[15 8 13];
>> GEshow(A,b')
Begin forward elimination with Augmented system:
      4     -2       1     15
     -3     -1       4      8
      1     -1       3     13
After elimination in column 1 with pivot = 4.000000
    4.0000   -2.0000    1.0000   15.0000
         0   -2.5000    4.7500   19.2500
         0   -0.5000    2.7500    9.2500
After elimination in column 2 with pivot = -2.500000
    4.0000   -2.0000    1.0000   15.0000
         0   -2.5000    4.7500   19.2500
         0         0    1.8000    5.4000
ans =
    2.0000
   -2.0000
    3.0000
```

Gaussian Elimination IV

- We shall obtain answers that are just close approximations to the exact answer because of round-off error.



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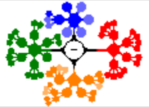
- When there are many equations, the effects of round-off (the term is applied to the error due to chopping as well as when rounding is used) may cause large effects.
- In certain cases, the coefficients are such that the results are particularly sensitive to round off; such systems are called **ill-conditioned**.

Gaussian Elimination V

- if we had stored the ratio of coefficients in place of zero (we show these in parentheses), our final form would have been

$$\begin{bmatrix} 4 & -2 & 1 & 15 \\ (-0.75) & -2.5 & 4.75 & 19.25 \\ (0.25) & (0.20) & 1.8 & 5.40 \end{bmatrix}$$



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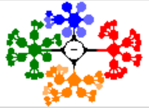

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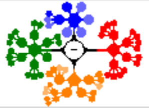

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- This procedure is called a <u>*LU-decomposition of A*</u>.
- In this case,

$$A = L * U$$

where L is lower-triangular and U is upper-triangular.

Gaussian Elimination VI

- The determinant of two matrices, $B \times C$, is the product of each of the determinants, for this example we have

$$det(L * U) = det(L) * det(U) = det(U)$$

Because L is triangular and has only ones on its diagonal so that $det(L) = 1$.

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- When there are row interchanges

$$det(A) = (-1)^m * u_{11} * \ldots * u_{nn}$$

  where the exponent m represents *the number of row interchanges*.

Gaussian Elimination VII

- **Example**. Solve the following system of equations using Gaussian elimination.

$$\begin{array}{rrrrl} & 2x_2 & & +x_4 & = 0 \\ 2x_1 & +2x_2 & +3x_3 & +2x_4 & = -2 \\ 4x_1 & -3x_2 & & x_4 & = -7 \\ 6x_1 & +x_2 & -6x_3 & -5x_4 & = 6 \end{array}$$

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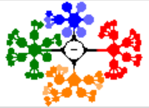

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Gaussian Elimination VIII

4 We again interchange before reducing the second column, not because we have a zero divisor, but because we want to preserve accuracy. Interchanging the second and third rows puts the element of largest magnitude on the diagonal.

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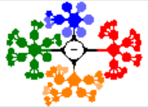
7 Back-substitution gives

$$x_1 = -0.50000, x_2 = 1.0000, x_3 = 0.33325, x_4 = -1.9999.$$



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- **Example m-files:** (GEshow.m, GEPivShow.m)

```
>> A=[0 2 0 1; 2 2 3 2; 4 -3 0 1; 6 1 -6 -5];
>> b=[0 -2 -7 6];
>> GEshow(A,b')
Begin forward elimination with Augmented system:
     0     2     0     1     0
     2     2     3     2    -2
     4    -3     0     1    -7
     6     1    -6    -5     6
??? Error using ==> GEshow
zero pivot encountered
>> GEpivShow(A,b')
```

Gaussian Elimination X

- In this example, if we had replaced the zeros below the main diagonal with the ratio of coefficients at each step, the resulting augmented matrix would be

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ (0.66667) & -3.6667 & 4 & 4.3333 & -11 \\ (0.33333) & (-0.45454) & 6.8182 & 5.6364 & -9.0001 \\ (0.0) & (-0.54545) & (0.32) & 1.5600 & -3.1199 \end{bmatrix}$$

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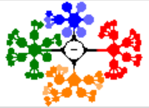

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- It should be noted that the product of these matrices produces a permutation of the original matrix, call it A', where

$$A' = \begin{bmatrix} 6 & 1 & -6 & -5 \\ 4 & -3 & 0 & 1 \\ 2 & 2 & 3 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

Gaussian Elimination XI

- The determinant of the original matrix of coefficients can be easily computed according to the formula

$$det(A) = (-1)^2 * (6) * (-3.6667) * (6.8182) * (1.5600) = -234.0028$$

which is close to the exact solution: -234.

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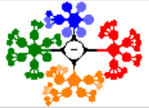

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Gaussian Elimination XI

- The determinant of the original matrix of coefficients can be easily computed according to the formula

$$det(A) = (-1)^2 * (6) * (-3.6667) * (6.8182) * (1.5600) = -234.0028$$

which is close to the exact solution: -234.

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- To summarize
  1. The solution to the four equations
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- "These" are readily obtained after solving the system by Gaussian elimination method.

Gaussian Elimination XII

Example m-file: LU factorization without pivoting. (luNopiv.m)

Example m-file: LU factorization with partial pivoting. (luPiv.m)

```
>> A=[0 2 0 1; 2 2 3 2; 4 -3 0 1; 6 1 -6 -5];
>> [L, U, P]=lu(A)
>> b = [ 0 -2 -7 6]
>> A\b
ans =
-0.5000
 1.0000
 0.3333
-2.0000
Remember
LUx=b
Ly=b --> find y first
Ux=y --> then find x
>> y=L\(P*b')
y =
    6.0000
  -11.0000
   -9.0000
   -3.1200
>> x=U\y
x =
   -0.5000
    1.0000
    0.3333
   -2.0000
```

The Gauss-Jordan Method I

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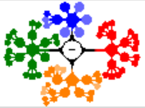

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- When this has been accomplished, the *column of right-hand sides* has been transformed into the solution vector.
- Pivoting is normally employed to preserve arithmetic accuracy.

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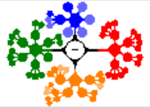
$$



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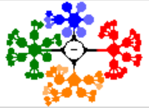
$$(1)\begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{bmatrix} \quad (2)\begin{bmatrix} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 0 & 1.6667 & 5 & 3.3667 & -4 \\ 0 & -3.6667 & 4 & 4.3334 & -11 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$



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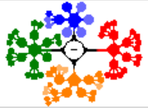

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2. Interchanging rows 2 and 3, dividing the new second row by -3.6667, and reducing the second column (operating above the diagonal as well as below) gives

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3. No interchanges now are required. We divide the third row by 6.8182 and create zeros below and above.

The Gauss-Jordan Method III

4 We complete by dividing the fourth row by 1.5599 and create zeros above.

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- While the Gauss-Jordan method might seem to require the same effort as Gaussian elimination, it really requires almost 50% more operations.