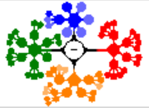

Lecture 6

Solving Sets of Equations II

Ill-Conditioned Systems, Iterative Methods

Ceng375 *Numerical Computations* at November 25, 2010

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Computer Engineering Department
Çankaya University

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Solving Sets of Equations

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- **Iterative Methods.** Describes how a linear system can be solved in an entirely different way, by beginning with an initial estimate of the solution and performing computations that eventually arrive at the correct solution. An iterative method is particularly important in solving systems that have few nonzero coefficients.

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- it is inefficient to solve the system from the start with each one of the right-hand-side values.
- Using the *LU* equivalent of the coefficient matrix is preferred.

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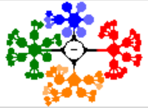

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- Which is again readily done by back-substitution (U is upper-triangular).

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- and then use this y in $Ux = y$ to find the new x:

$$x = [0.0128\ -0.5897\ -2.0684\ 2.1795]^T$$

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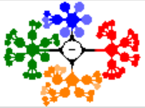

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The Inverse of a Matrix and Matrix Pathology II

- Cont.

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The Inverse of a Matrix and Matrix Pathology II

- Cont.

$$\begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & 0 & 1 & -3 \end{bmatrix}, \qquad {\color{red} R_3/(-5)} \; ,$$

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$$\left[\begin{array}{cccccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & 0 & 1 & -3 \end{array}\right], \quad \begin{array}{c} \\ R_3/(-5) \end{array}, \quad \begin{array}{c} R_1 - (2/1)R_3 \rightarrow \\ \\ \end{array},$$

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The Inverse of a Matrix and Matrix Pathology II

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$$\begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & 0 & 1 & -3 \end{bmatrix}, \quad \begin{matrix} \\ R_3/(-5) \end{matrix}, \quad \begin{matrix} R_1 - (2/1)R_3 \rightarrow \\ \\ \end{matrix},$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 2/5 & -6/5 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1/5 & 3/5 \end{bmatrix}, \quad R_2 - (1/-1)R_1 \rightarrow ,$$

The Inverse of a Matrix and Matrix Pathology II

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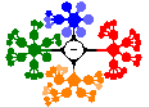

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The Inverse of a Matrix and Matrix Pathology II

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The Inverse of a Matrix and Matrix Pathology II

- Cont.

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The Inverse of a Matrix and Matrix Pathology III

- It is more *efficient* to use Gaussian elimination. We show only the final triangular matrix; we used pivoting:

$$\begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & 1 & 0 & 1 & 0 \\ (0.333) & -1 & 1.667 & 1 & -0.333 & 0 \\ (0.333) & (0) & 1.667 & 0 & -0.333 & 1 \end{bmatrix}$$

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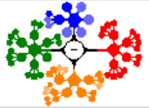
- After doing the back-substitutions, we get

$$\left[\begin{array}{rrrrrr} 3 & 0 & 1 & 0 & 0.4 & -0.2 \\ (0.333) & -1 & 1.667 & -1 & 0 & 1 \\ (0.333) & (0) & 1.667 & 0 & -0.2 & 0.6 \end{array}\right]$$



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- If we have the inverse of a matrix, we can *use it to solve a set of equations*, $Ax = b$,
- because multiplying by A^{-1} gives the answer (x):

$$A^{-1}Ax = A^{-1}b$$
$$x = A^{-1}b$$

Pathological Systems

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- Here is an example of a matrix that has no inverse:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -1 \\ -1 & -14 & 11 \end{bmatrix}$$

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>> lu(A)
ans =
 2.0000      4.0000     -1.0000
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- The definition of a *singular matrix* is a matrix that does not have an inverse.

Redundant Systems I

- Even though a matrix is singular, it may still have a solution. Consider again the same singular matrix:

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  - We have found a solution, actually, an infinity of them. The reason for this is that the system is **redundant**.

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- See Table 1 for the comparison of *singular* and *nonsingular* matrices.

Table: A comparison of singular and nonsingular matrices

For Singular matrix A:	For Nonsingular Matrix A:
It has no inverse, A^{-1}	It has an inverse, A^{-1}
Its determinant is zero	The determinant is nonzero
There is no unique solution	There is a unique solution
to the system $Ax = b$	to the system $Ax = b$
Gaussian elimination cannot avoid	Gaussian elimination does not
a zero on the diagonal	encounter a zero on the diagonal
The rank is less than n	The rank equals n
Rows are linearly dependent	Rows are linearly independent
Columns are linearly dependent	Columns are linearly independent

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- The LU equivalent has a very small element in (3, 3),

$$LU = \begin{bmatrix} 4.33 & 0.56 & -1.78 \\ (0.6975) & -1.4406 & 3.7715 \\ (-0.1917) & (0.3003) & -0.0039 \end{bmatrix},$$

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- Inverse has elements very large in comparison to A:

$$inv(A) = \begin{bmatrix} 5.6611 & -7.2732 & -18.5503 \\ 200.5046 & -268.2570 & -66669.9143 \\ 76.8511 & -102.6500 & -255.8846 \end{bmatrix}$$

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- Matrix is nonsingular but is *almost* singular.

Ill-Conditioned Systems II

- Suppose we solve the system $Ax = b$, with b equal to $[-1.61, 7.23, -3.38]^T$.

```
>> A=[3.02 -1.05 2.53; 4.33 0.56 -1.78; -0.83 -0.54 1.47];
b=[-1.61 7.23 -3.38];A\b'
ans =
    1.0000
    2.0000
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>> A=[3.02 -1.05 2.53; 4.33 0.56 -1.78; -0.83 -0.54 1.47];
b=[-1.60 7.23 -3.38];A\b'
ans =
    1.0566
    4.0050
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>> A=[3.02 -1.05 2.53; 4.33 0.56 -1.78; -0.83 -0.54 1.47];
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    1.0727
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    - We get $x = [1.0566, 4.0051, -0.2315]^T$

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- Now suppose that we make a **small change** in just the first element of the b-vector : $[-1.60, 7.23, -3.38]^T$.
  - We get $x = [1.0566, 4.0051, -0.2315]^T$
- if $b = [-1.61, 7.22, -3.38]^T$, the solution now is $x = [1.07271, 4.6826, 0.0265]^T$ which also differs.

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>> A=[3.02 -1.05 2.53; 4.33 0.56 -1.78; -0.83 -0.54 1.47];
b=[-1.61 7.22 -3.38];A\b'
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    1.0727
    4.6826
    0.0265
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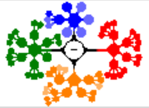

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- This means that it is also very sensitive to round-off error.

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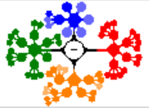

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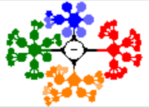

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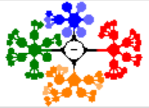

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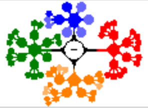

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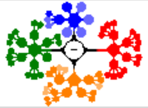

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- i.e., Compute the 1-, 2-, and ∞-norms of the vector $x = (1.25, 0.02, -5.15, 0)$

$$||x||_1 = |1.25| + |0.02| + |-5.15| + |0| = 6.42$$
$$||x||_2 = 5.2996$$
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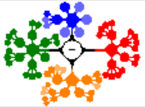

Matrix Norms II

```
A =
   5 -5 -7
  -4  2 -4
  -7 -4  5
>> norm(A,'fro')
ans =
15
>> norm(A,inf)
ans =
17
>> norm(A,1)
ans=
16
>> norm(A)
ans =
12.0301
>> norm (A,2)
ans =
12.0301
```

we observe that the 2-norm, the spectral norm, is the norm we get if we just ask for the norm. The smallest norm of the matrix is the spectral norm, it is the tightest measure.

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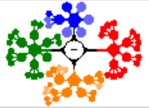

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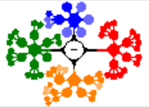

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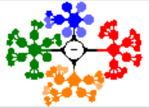

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- However, before we begin our iterative scheme we must first reorder the equations so that the coefficient matrix is diagonally dominant.

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- After reordering;

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- The iterative methods depend on the rearrangement of the equations in this manner:

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- and the process is repeated until successive values of each of the variables are sufficiently alike.

Jacobi Method II

- Note that this method is exactly the same as the method of fixed-point iteration for a single equation that was discussed in Section **??**.
- But it is now applied to a set of equations; we see this if we write Eqn. 1 in the form of

$$x^{(n+1)} = G(x^{(n)}) = b' - Bx^{n}$$

which is identical to $x_{n+1} = g(x_n)$ as used in Section **??**.

- The new values are substituted in the right-hand sides to generate a second approximation,
- and the process is repeated until successive values of each of the variables are sufficiently alike.
- Now, general form

$$\begin{aligned} x_1^{(n+1)} &= 1.8333 + 0.3333x_2^{(n)} - 0.1667x_3^{(n)} \\ x_2^{(n+1)} &= 0.7143 + 0.2857x_1^{(n)} - 0.2857x_3^{(n)} \\ x_3^{(n+1)} &= 0.2000 + 0.2000x_1^{(n)} + 0.4000x_2^{(n)} \end{aligned} \qquad (2)$$

Jacobi Method III

- Starting with an initial vector of $x^{(0)} = (0, 0, 0,)$, we obtain Table 2

	First	Second	Third	Fourth	Fifth	Sixth	...	Ninth
x_1	0	1.833	2.038	2.085	2.004	1.994	...	2.000
x_2	0	0.714	1.181	1.053	1.001	0.990	...	1.000
x_3	0	0.200	0.852	1.080	1.038	1.001	...	1.000

Table: Successive estimates of solution (Jacobi method)

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- In the present context, $x^{(n)}$ and $x^{(n+1)}$ refer to the n^{th} and $(n+1)^{st}$ iterates of a vector rather than a simple variable, and g is a linear transformation rather than a nonlinear function.

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- Rewrite in matrix notation; let $A = L + D + U$,

$$Ax = b, \begin{bmatrix} 6 & -2 & 1 \\ -2 & 7 & 2 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}, D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -5 \end{bmatrix}, U = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Jacobi Method IV

$$
\begin{aligned}
Ax &= (L + D + U)x = b \\
Dx &= -(L + U)x + b \\
x &= -D^{-1}(L + U)x + D^{-1}b
\end{aligned}
$$

- From this we have, identifying x on the left as the new iterate,

$$x^{(n+1)} = -D^{-1}(L + U)x^{(n)} + D^{-1}b$$

In Eqn. 2,

$$b' = D^{-1}b = \begin{bmatrix} 1.8333 \\ 0.7143 \\ 0.2000 \end{bmatrix}$$

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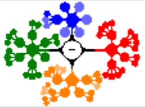
- This procedure is known as the Jacobi method, also called "*the method of simultaneous displacements*",



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- This procedure is known as the Jacobi method, also called "*the method of simultaneous displacements*",
- because each of the equations is simultaneously changed by using the most recent set of x-values (see Table 2).

Gauss-Seidel Iteration I

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Gauss-Seidel Iteration I

- Even though we have *newx*[1] available, we do not use it to compute *newx*[2].
- In nearly all cases the new values are better than the old and ought to be used instead.
- When this done, the procedure known as *Gauss-Seidel* iteration.
- We proceed to improve each x-value in turn, using always the most recent approximations of the other variables.

	First	Second	Third	Fourth	Fifth	Sixth
x_1	0	1.833	2.069	1.998	1.999	2.000
x_2	0	1.238	1.002	0.995	1.000	1.000
x_3	0	1.062	1.015	0.998	1.000	1.000

Table: Successive estimates of solution (Gauss-Seidel method)

Gauss-Seidel Iteration II

- These values were computed by using this iterative scheme:

$$
\begin{aligned}
x_1^{(n+1)} &= 1.8333 + 0.3333x_2^{(n)} - 0.1667x_3^{(n)} \\
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  beginning with $x^{(1)} = (0, 0, 0)^T$

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- The rate of convergence is more rapid than for the Jacobi method (see Table 3).