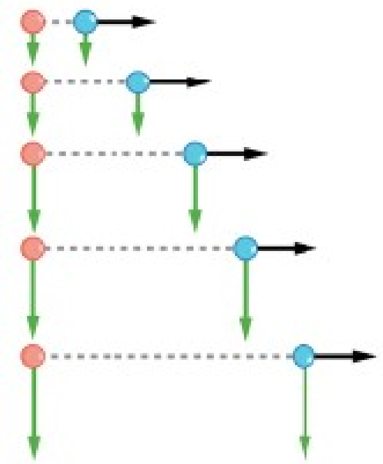
Vectors

- Quantities which indicate both magnitude and direction
- Examples: displacement, velocity, acceleration

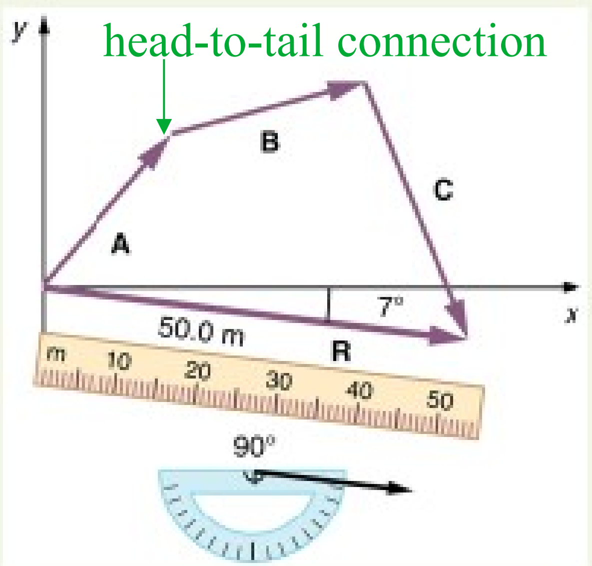
Scalars

- Quantities which indicate only magnitude
- Examples: Time, speed, temperature, distance



Chapter 3

Vectors



3 VECTORS 38

relatively complicated

easier way

3-2 Vectors and Scalars

- Physics deals with many quantities that have both **size** and **direction.**
- It needs a *special mathematical language*-the language of vectors-to describe those quantities. (in magnitude, in direction)
- In this chapter: we will focus on the basic language of vectors.

Vectors

- Quantities which indicate both magnitude and direction
- Examples: displacement, velocity, acceleration

Scalars

- Quantities which indicate only magnitude
- Examples: Time, speed, temperature, distance

- The simplest example is a **displacement vector.**
- If a particle changes position from A to B, we represent this by a vector arrow pointing from A to B.

  A tail head B $\to x$

  x_i x_f
- Arrows are used to represent vectors.

  magnitude
  - The **length** of the arrow signifies **magnitude**
  - The **head** of the arrow signifies **direction**

  need a reference coordinate system (R.H)

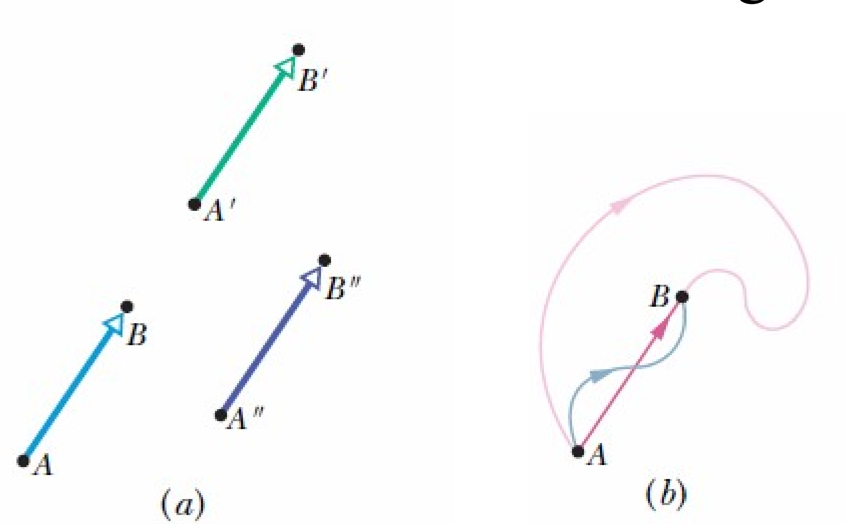


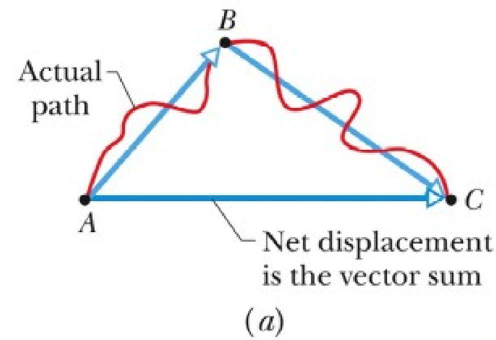
Fig. 3-1 (*a*) All three arrows have the same magnitude and direction and thus represent the same displacement. (*b*) All three paths connecting the two points correspond to the same displacement vector.

- Sometimes the vectors are represented by bold lettering, such as vector **a**.
- Sometimes they are represented with arrows on the top, such as $\vec{A}$

Vector Algebra

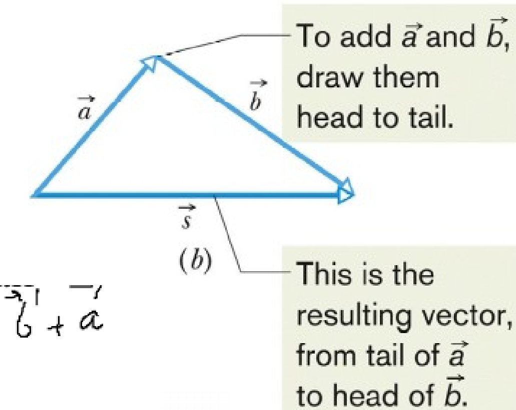
- The **vector sum**, or **resultant**
  - Is the result of performing vector addition
  - Represents the *net displacement* of two or more displacement vectors

initial $\rightarrow$ final
no intermediate steps



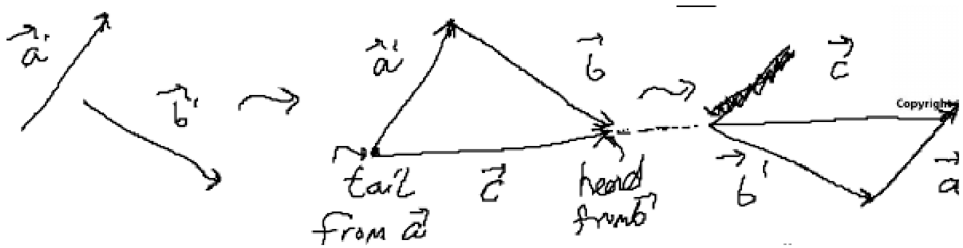
- Vector **a** and vector **b** can be added geometrically to yield the resultant vector sum, **s**.

$$\vec{s} = \vec{a} + \vec{b},$$



•Place the second vector, **b**, with its tail touching the head of the first vector, **a**.
•The vector sum, **s**, is the vector joining the tail of **a** to the head of **b**.

$\vec{a}+\vec{b}=\vec{c}=\vec{b}+\vec{a}$

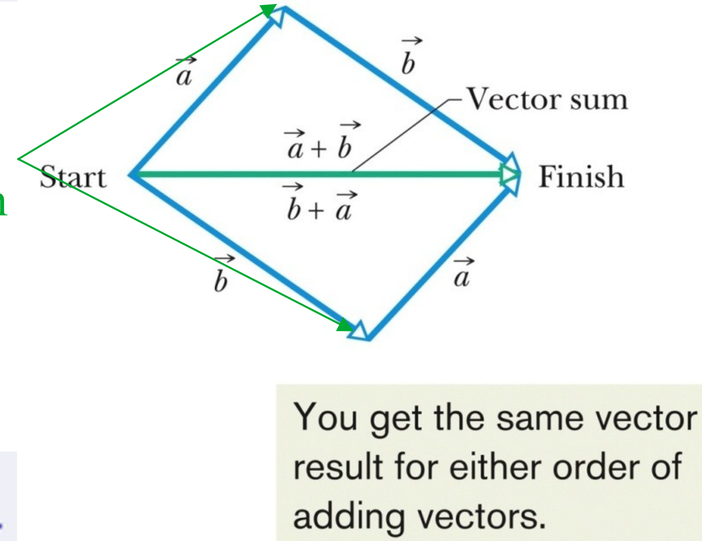


3-3 Adding vectors geometrically; Some rules

- Vector addition is **commutative**
  - We can add vectors in any order

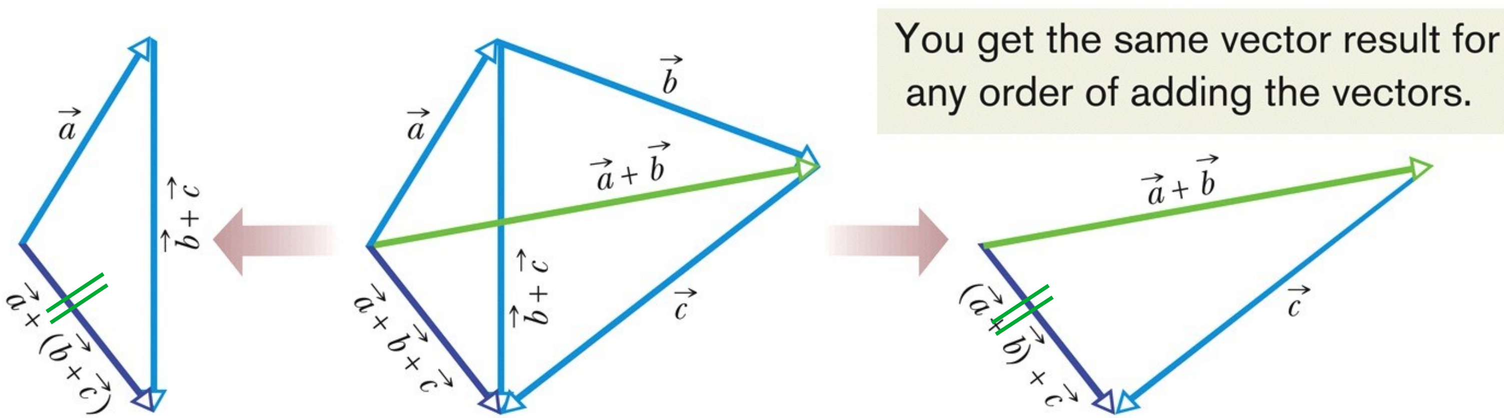
$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \text{(commutative law).}$$

head-tail connection



- Vector addition is **associative**
- We can group vector addition however we like

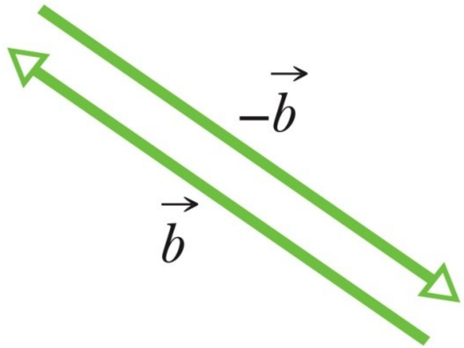
$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \qquad \text{(associative law).}$$



3-3 Adding vectors geometrically; Some rules

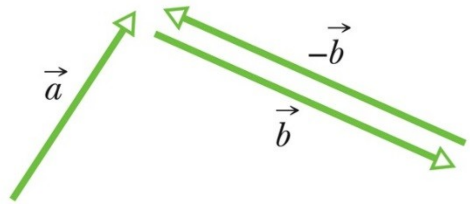
- A negative sign reverses vector direction

$$\vec{b} + (-\vec{b}) = 0.$$

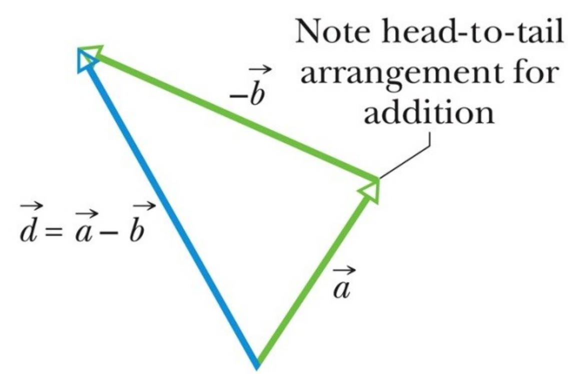


- We use this to define vector subtraction

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b}) \qquad \text{(vector subtraction}$$



(a)



(b)

- Only vectors of the same kind can be added
  - (*distance*) + (*distance*) makes sense
  - (*distance*) + (*velocity*) does not

Example:

In an orienteering class, you have the goal of moving as far (straight-line distance) from base camp as possible by making three straight-line moves. You may use the following displacements in any order: (a) $\vec{a}$, 2.0 km due east (directly toward the east); (b) $\vec{b}$, 2.0 km 30° north of east (at an angle of 30° toward the north from due east); (c) $\vec{c}$, 1.0 km due west. Alternatively, you may substitute either $-\vec{b}$ for $\vec{b}$ or $-\vec{c}$ for $\vec{c}$. What is the greatest distance you can be from base camp at the end of the third displacement?

Reasoning: Using a convenient scale, we draw vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $-\vec{b}$, and $-\vec{c}$ as in Fig. 3-7*a*. We then mentally slide the vectors over the page, connecting three of them at a time in head-to-tail arrangements to find their vector sum $\vec{d}$. The tail of the first vector represents base camp. The head of the third vector represents the point at which you stop. The vector sum $\vec{d}$ extends from the tail of the first vector to the head of the third vector. Its magnitude d is your distance from base camp.

We find that distance d is greatest for a head-to-tail arrangement of vectors $\vec{a}$, $\vec{b}$, and $-\vec{c}$. They can be in any order, because their vector sum is the same for any order.

$\vec{b} + \vec{a} + (-\vec{c}) = \vec{d}$

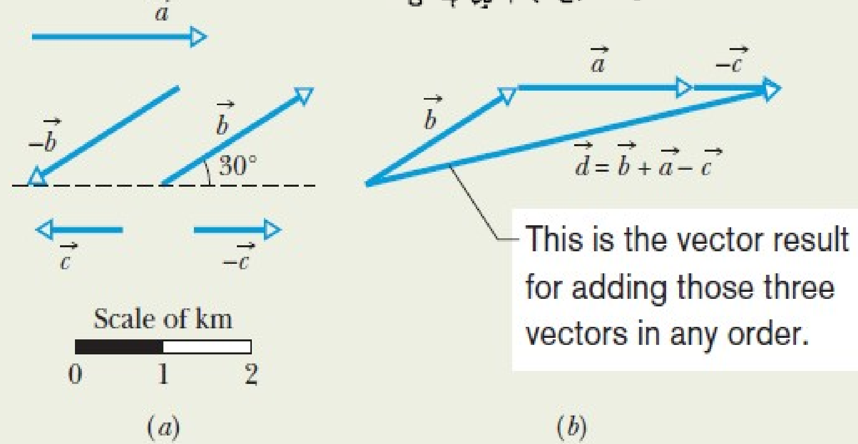


Fig. 3-7 (*a*) Displacement vectors; three are to be used. (*b*) Your distance from base camp is greatest if you undergo displacements $\vec{a}$, $\vec{b}$, and $-\vec{c}$, in any order.

The order shown in Fig. 3-7*b* is for the vector sum

$$\vec{d} = \vec{b} + \vec{a} + (-\vec{c}).$$

Using the scale given in Fig. 3-7*a*, we measure the length d of this vector sum, finding

$$d = 4.8 \text{ m}. \qquad \text{(Answer)}$$

magnitude

3-4 Components of vectors

- The component of a vector along an axis is the projection of the vector onto that axis.

- The process of finding the components of a vector is called *resolution of the vector*. $A_x \hat{x} + A_y \hat{y} = \vec{A}$
- In 3-dimensions, there are three components of a vector along pre-defined x-, y-, and z-axes.

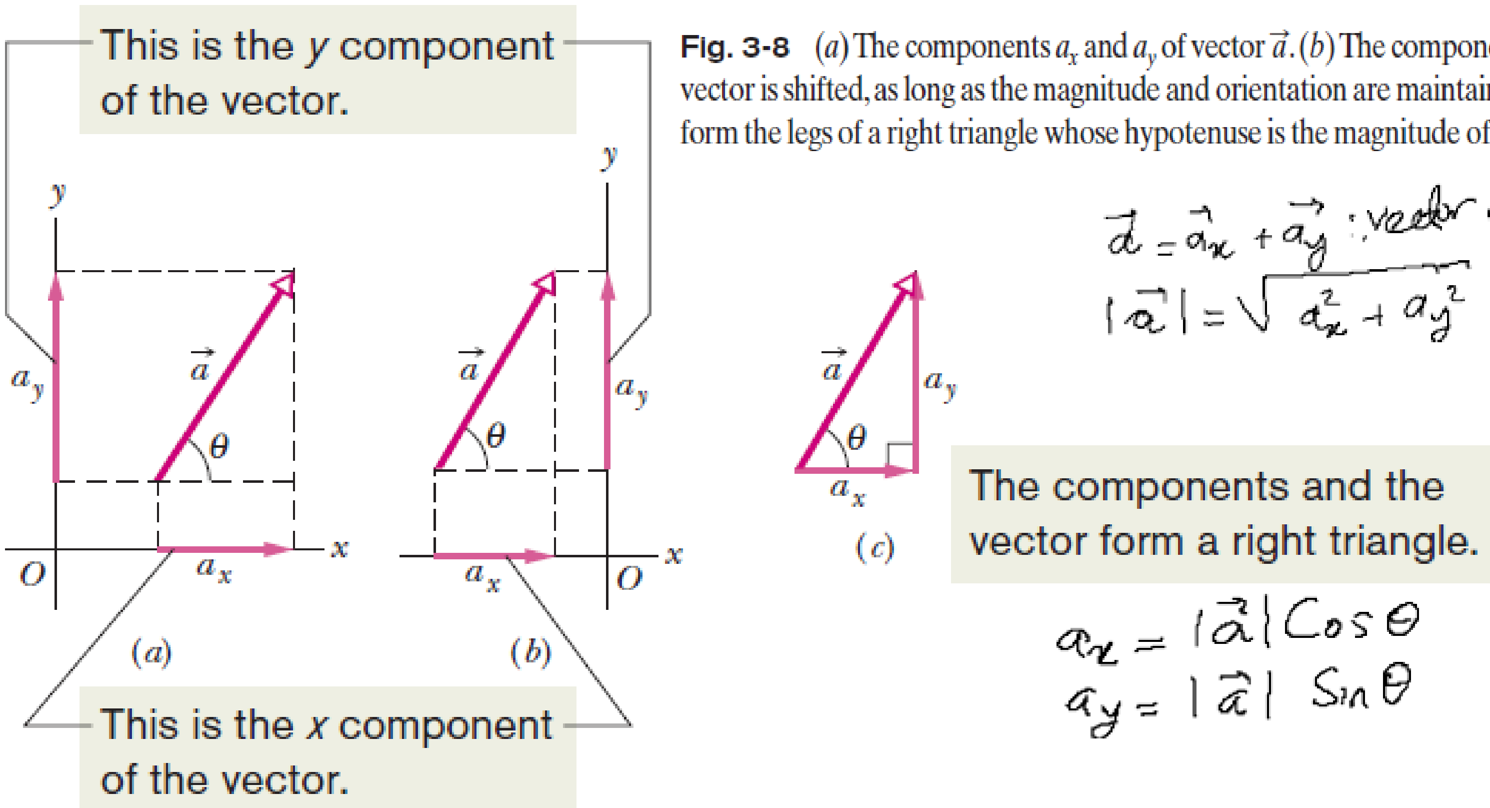


Fig. 3-8 (*a*) The components a_x and a_y of vector $\vec{a}$. (*b*) The components are unchanged if the vector is shifted, as long as the magnitude and orientation are maintained. (*c*) The components form the legs of a right triangle whose hypotenuse is the magnitude of the vector.

$\vec{a} = \vec{a}_x + \vec{a}_y$: vector algebra

$|\vec{a}| = \sqrt{a_x^2 + a_y^2}$

$a_x = |\vec{a}| \cos\theta$

$a_y = |\vec{a}| \sin\theta$

- The components of a vector can be positive or negative.
- We find the components of a vector by using the right triangle rules.

ccw: counter clock wise

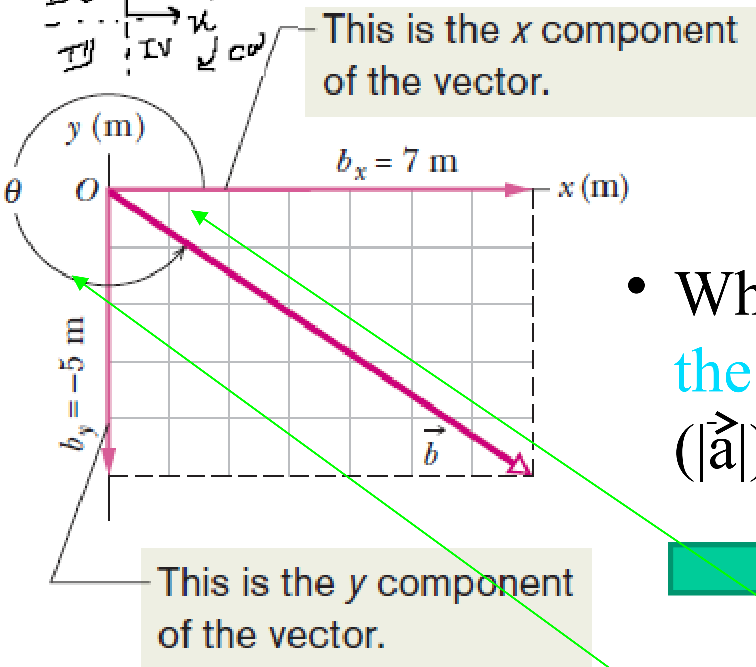


Fig. 3-9 The component of $\vec{b}$ on the x axis is positive, and that on the y axis is negative.

→ (+) +30° ccw

−30° cw

$$a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta,$$

- Where θ is the angle the vector makes with the positive x axis, and a is the vector length ($|\vec{a}|$).

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\theta = \frac{a_y}{a_x}$$

cw $\theta = \arctan -\frac{5}{7}$

ccw → 360° − θ

do not forget the signs (±)

- They (a_x & a_y) are unchanged if the vector is shifted in any direction (but not rotated).

arc → length of an curvature → small angle → approximation

3-4 Components of vectors

- Angles may be measured in degrees or radians
- Recall that a full circle is 360˚, or 2π rad

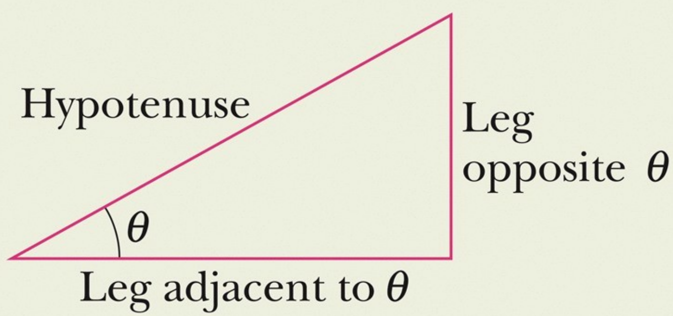
$$40^\circ \frac{2\pi \text{ rad}}{360^\circ} = 0.70 \text{ rad.}$$

- Know the three basic trigonometric functions

$$\sin\theta = \frac{\text{leg opposite } \theta}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{leg adjacent to } \theta}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{leg opposite } \theta}{\text{leg adjacent to } \theta}$$



A small airplane leaves an airport on an overcast day and is later sighted 215 km away, in a direction making an angle of 22° east of due north. How far east and north is the airplane from the airport when sighted?

Example

N
W E
S

KEY IDEA

We are given the magnitude (215 km) and the angle (22° east of due north) of a vector and need to find the components of the vector.

$\theta \rightarrow 90° - 22° = 68°$

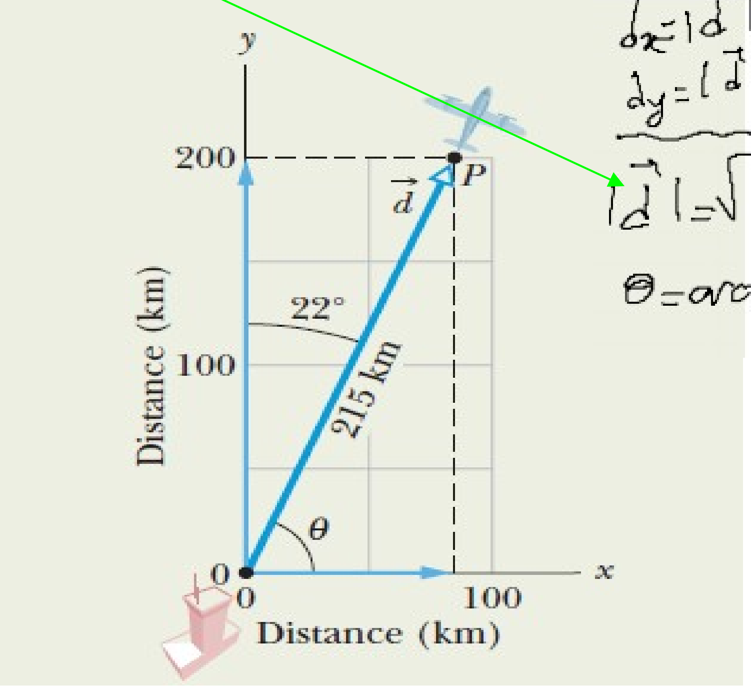
$d_x = |\vec{d}| \cos 68°$

$d_y = |\vec{d}| \sin 68°$

$|\vec{d}| = \sqrt{d_x^2 + d_y^2}$

$\theta = \arctan \frac{d_y}{d_x}$

Calculations: We draw an xy coordinate system with the positive direction of x due east and that of y due north (Fig. 3-10). For convenience, the origin is placed at the airport. The airplane's displacement $\vec{d}$ points from the origin to where the airplane is sighted.

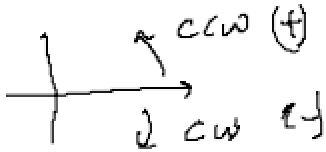


To find the components of $\vec{d}$, we use Eq. 3-5 with $\theta = 68°$ $(= 90° - 22°)$:

$$d_x = d \cos \theta = (215 \text{ km})(\cos 68°)$$
$$= 81 \text{ km} \qquad \text{(Answer)}$$

$$d_y = d \sin \theta = (215 \text{ km})(\sin 68°)$$
$$= 199 \text{ km} \approx 2.0 \times 10^2 \text{ km.} \qquad \text{(Answer)}$$

Thus, the airplane is 81 km east and 2.0×10^2 km north of the airport.



$(\tan^{-1})\tan\theta$

inverse function

$\theta = \tan^{-1}\frac{y}{x}$

Angles measured counterclockwise will be considered positive, and clockwise negative.

Change the units of the angles to be consistent.

Use definitions of trig functions and inverse trig functions to find components.

Check calculator results.

Check if the angles are measured counterclockwise from the positive direction of the x-axis, in which case the angles will be positive.

- A **unit vector**
  - Has magnitude 1
  - Has a particular direction
  - Lacks both dimension and unit
  - Is labeled with a hat: ^

$a_x \hat{x} + a_y \hat{y}$

$a_x \hat{i} + a_y \hat{j}$

$|\hat{x}| = 1$

- Unit vectors pointing in the x-, y-, and z-axes are usually designated by $\hat{i}, \hat{j}, \hat{k}$ respectively

- Therefore vector, $\vec{a}$, with components a_x and a_y in the x- and y-directions, can be written in terms of the following vector sum:

$$\vec{a} = a_x\hat{i} + a_y\hat{j}$$

- We use a **right-handed coordinate system**

- Remains right-handed when rotated

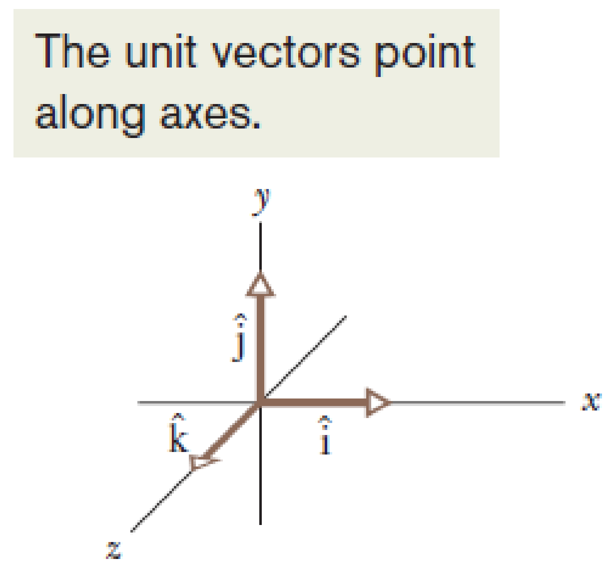


Fig. 3-13 Unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ define the directions of a right-handed coordinate system.

3-6 Adding vectors by components

If $\boxed{\vec{r} = \vec{a} + \vec{b},}$ then $\boxed{\begin{aligned} r_x &= a_x + b_x \\ r_y &= a_y + b_y \\ r_z &= a_z + b_z. \end{aligned}}$

no head-tail connections, but component wise.

•Two vectors must be equal if their corresponding components are equal.

•The procedure of adding vectors also applies to vector subtraction.

Therefore, $\boxed{\vec{d} = \vec{a} - \vec{b}}$ $\Rightarrow$ $\boxed{d_x = a_x - b_x, \quad d_y = a_y - b_y, \quad \text{and} \quad d_z = a_z - b_z,}$

where $\vec{d} = d_x\hat{\mathrm{i}} + d_y\hat{\mathrm{j}} + d_z\hat{\mathrm{k}}.$

Example:

The desert ant *Cataglyphis fortis* lives in the plains of the Sahara desert. When one of the ants forages for food, it travels from its home nest along a haphazard search path, over flat, featureless sand that contains no landmarks. Yet, when the ant decides to return home, it turns and then runs directly home. According to experiments, the ant keeps track of its movements along a mental coordinate system. When it wants to return to its home nest, it effectively sums its displacements along the axes of the system to calculate a vector that points directly home. As an example of the calculation, let's consider an ant making five runs of 6.0 cm each on an xy coordinate system, in the directions shown in Fig. 3-16a, starting from home. At the end of the fifth run, what are the magnitude and angle of the ant's net displacement vector $\vec{d}_{\text{net}}$, and what are those of the homeward vector $\vec{d}_{\text{home}}$ that extends from the ant's final position back to home? In a real situation, such vector calculations might involve thousands of such runs.

KEY IDEAS

(1) To find the net displacement $\vec{d}_{\text{net}}$, we need to sum the five individual displacement vectors:

$$\vec{d}_{\text{net}} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3 + \vec{d}_4 + \vec{d}_5.$$

(2) We evaluate this sum for the x components alone,

$$d_{\text{net},x} = d_{1x} + d_{2x} + d_{3x} + d_{4x} + d_{5x},$$

and for the y components alone,

$$d_{\text{net},y} = d_{1y} + d_{2y} + d_{3y} + d_{4y} + d_{5y}.$$

(3) We construct $\vec{d}_{\text{net}}$ from its x and y components.

Calculations:

$$d_{1x} = (6.0 \text{ cm}) \cos 0° = +6.0 \text{ cm}$$
$$d_{2x} = (6.0 \text{ cm}) \cos 150° = -5.2 \text{ cm}$$
$$d_{3x} = (6.0 \text{ cm}) \cos 180° = -6.0 \text{ cm}$$
$$d_{4x} = (6.0 \text{ cm}) \cos(-120°) = -3.0 \text{ cm}$$
$$d_{5x} = (6.0 \text{ cm}) \cos 90° = 0.$$

$$d_{\text{net},x} = +6.0 \text{ cm} + (-5.2 \text{ cm}) + (-6.0 \text{ cm}) + (-3.0 \text{ cm}) + 0 = -8.2 \text{ cm}.$$

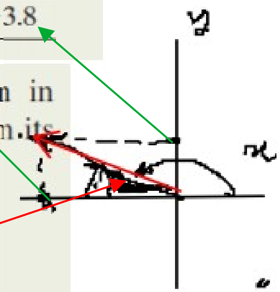
$$d_{\text{net},y} = +3.8 \text{ cm}.$$

TABLE 3-1

Run	d_x (cm)	d_y (cm)
1	+6.0	0
2	−5.2	+3.0
3	−6.0	0
4	−3.0	−5.2
5	0	+6.0
net	−8.2	+3.8

Vector $\vec{d}_{\text{net}}$ and its x and y components are shown in Fig. 3-16b. To find the magnitude and angle of $\vec{d}_{\text{net}}$ from its components, we use Eq. 3-6. The magnitude is

$$d_{\text{net}} = \sqrt{d_{\text{net},x}^2 + d_{\text{net},y}^2} = \sqrt{(-8.2 \text{ cm})^2 + (3.8 \text{ cm})^2} = 9.0 \text{ cm}.$$



180 − 24.86
ccw

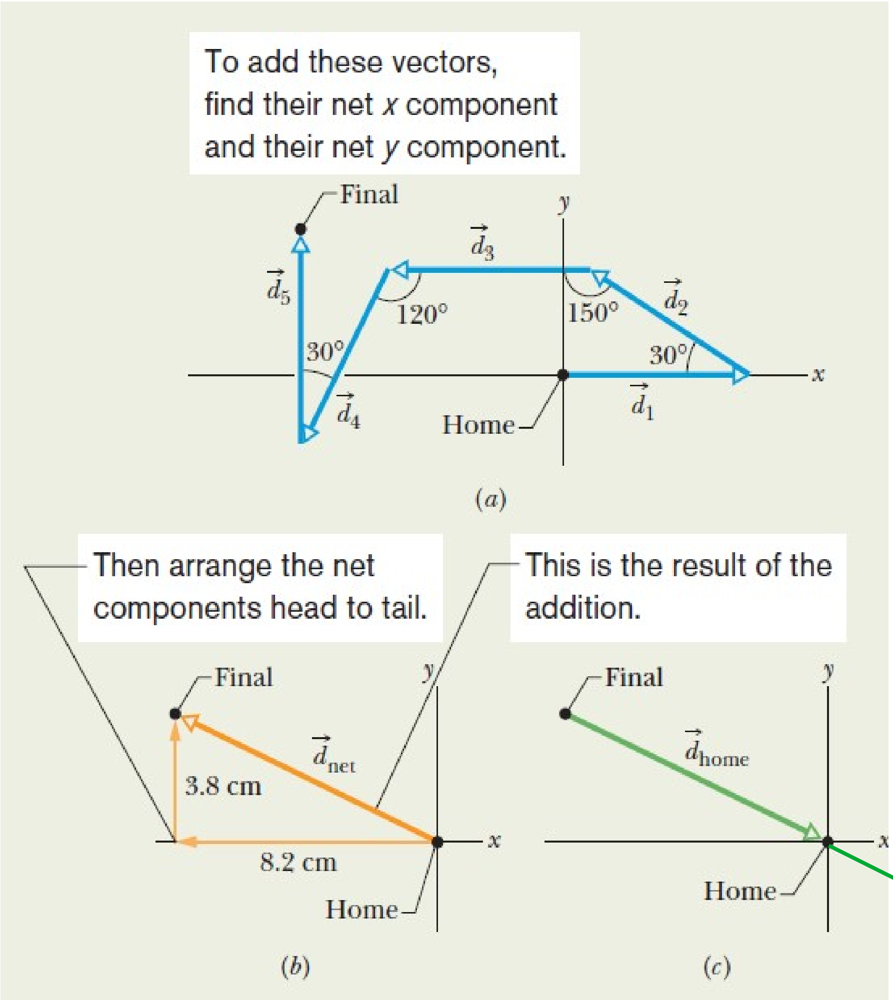
To find the angle (measured from the positive direction of x), we take an inverse tangent:

$$\theta = \tan^{-1}\left(\frac{d_{\text{net},y}}{d_{\text{net},x}}\right) = \tan^{-1}\left(\frac{3.8 \text{ cm}}{-8.2 \text{ cm}}\right) = -24.86°.$$

cw

do not forget the signs (+)

Example (Continued):



-25° cw

Caution: Taking an inverse tangent on a calculator may not give the correct answer. The answer −24.86° indicates that the direction of $\vec{d}_{net}$ is in the fourth quadrant of our *xy* coordinate system. However, when we construct the vector from its components (Fig. 3-16*b*), we see that the direction of $\vec{d}_{net}$ is in the second quadrant. Thus, we must "fix" the calculator's answer by adding 180°:

$$\boxed{\theta = -24.86° + 180° = 155.14° \approx 155°.}$$

Thus, the ant's displacement $\vec{d}_{net}$ has magnitude and angle

$$\boxed{d_{net} = 9.0 \text{ cm at } 155°.} \qquad \text{(Answer)}$$

Vector $\vec{d}_{home}$ directed from the ant to its home has the same magnitude as $\vec{d}_{net}$ but the opposite direction (Fig. 3-16*c*). We already have the angle (−24.86° ≈ −25°) for the direction opposite $\vec{d}_{net}$. Thus, $\vec{d}_{home}$ has magnitude and angle

$$\boxed{d_{home} = 9.0 \text{ cm at } -25°.} \qquad \text{(Answer)}$$

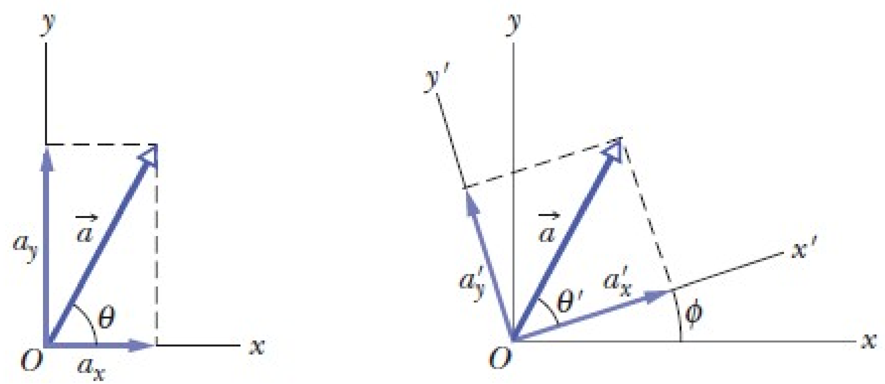
A desert ant traveling more than 500 m from its home will actually make thousands of individual runs. Yet, it somehow knows how to calculate $\vec{d}_{home}$ (without studying this chapter).

- Freedom of choosing a coordinate system.
- Relations among vectors do not depend on the origin or the orientation of the axes.
- Relations in physics are also independent of the choice of the coordinate system.
- We can rotate the coordinate system, without rotating the vector, and the vector remains the same.

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{a_x'^2 + a_y'^2}$$

$$\theta = \theta' + \phi.$$

Rotating the axes changes the components but not the vector.



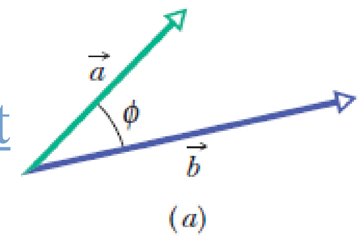
- All such coordinate systems are equally valid.

FYI

A. Multiplying a vector by a scalar

Multiplying a vector by a scalar changes the magnitude but not the direction: $\vec{a} * s = s\vec{a}$ 3x ⇒ triple in length

B. Multiplying a vector by a vector: Scalar (Dot) Product



The scalar product between two vectors is written as:

$$\vec{a} \cdot \vec{b}$$

It is defined as:

$$\vec{a} \cdot \vec{b} = ab \cos \phi,$$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}.$$ order is NOT important!

$|a \cos\phi||\vec{b}|$ } same
$|\vec{a}||b \cos\phi|$

- Here, a and b are the magnitudes of vectors **a** and **b** respectively, and φ is the angle between the two vectors.
- The right hand side is a scalar quantity.

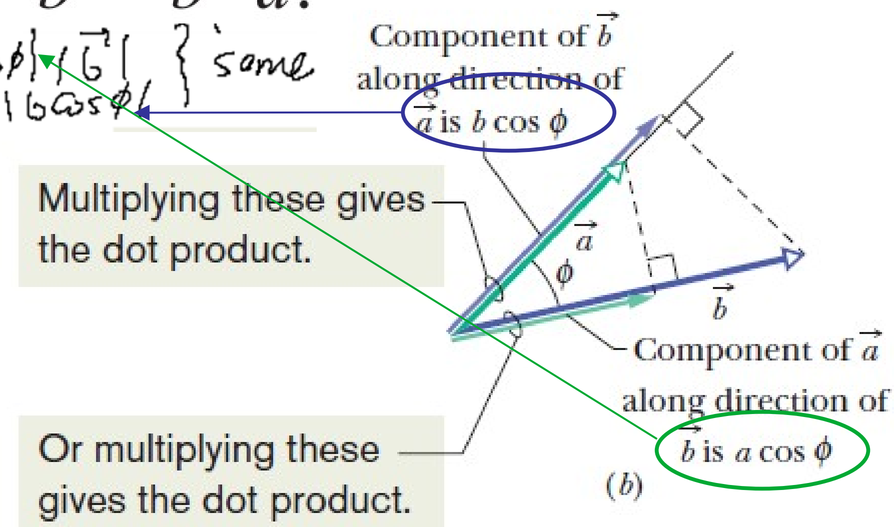


Fig. 3-18 (a) Two vectors $\vec{a}$ and $\vec{b}$, with an angle ϕ between them. (b) Each vector has a component along the direction of the other vector.

C. Multiplying a vector with a vector: Vector (Cross) Product

- The vector product between two vectors **a** and **b** can be written as: $\vec{a} \times \vec{b}$
- The result is a new vector **c**, which is: $c = ab \sin \phi,$
  - Here a and b are the magnitudes of vectors **a** and **b** respectively, and ϕ is the smaller of the two angles between **a** and **b** vectors.

The right-hand rule allows us to find the direction of vector **c**.

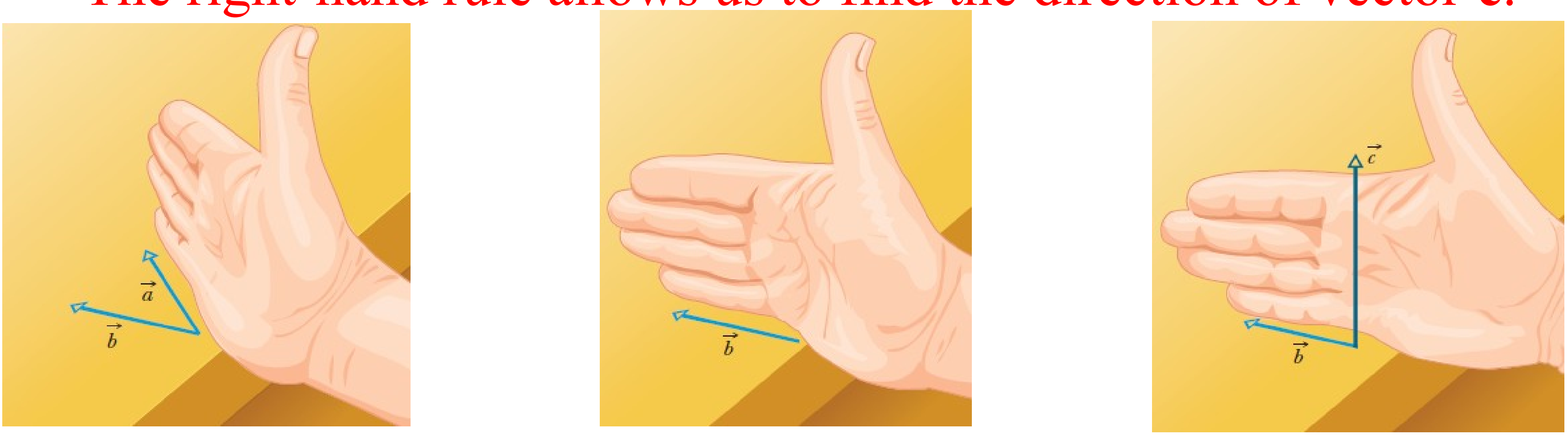


Fig. 3-19 Illustration of the right-hand rule for vector products. (*a*) Sweep vector $\vec{a}$ into vector $\vec{b}$ with the fingers of your right hand. Your outstretched thumb shows the direction of vector $\vec{c} = \vec{a} \times \vec{b}$.

$\vec{b} \times \vec{a}$ is same?
NOT exactly

$\vec{c}$ vector is not in the plane of $\vec{a}$ & $\vec{b}$ but $\perp$ to them

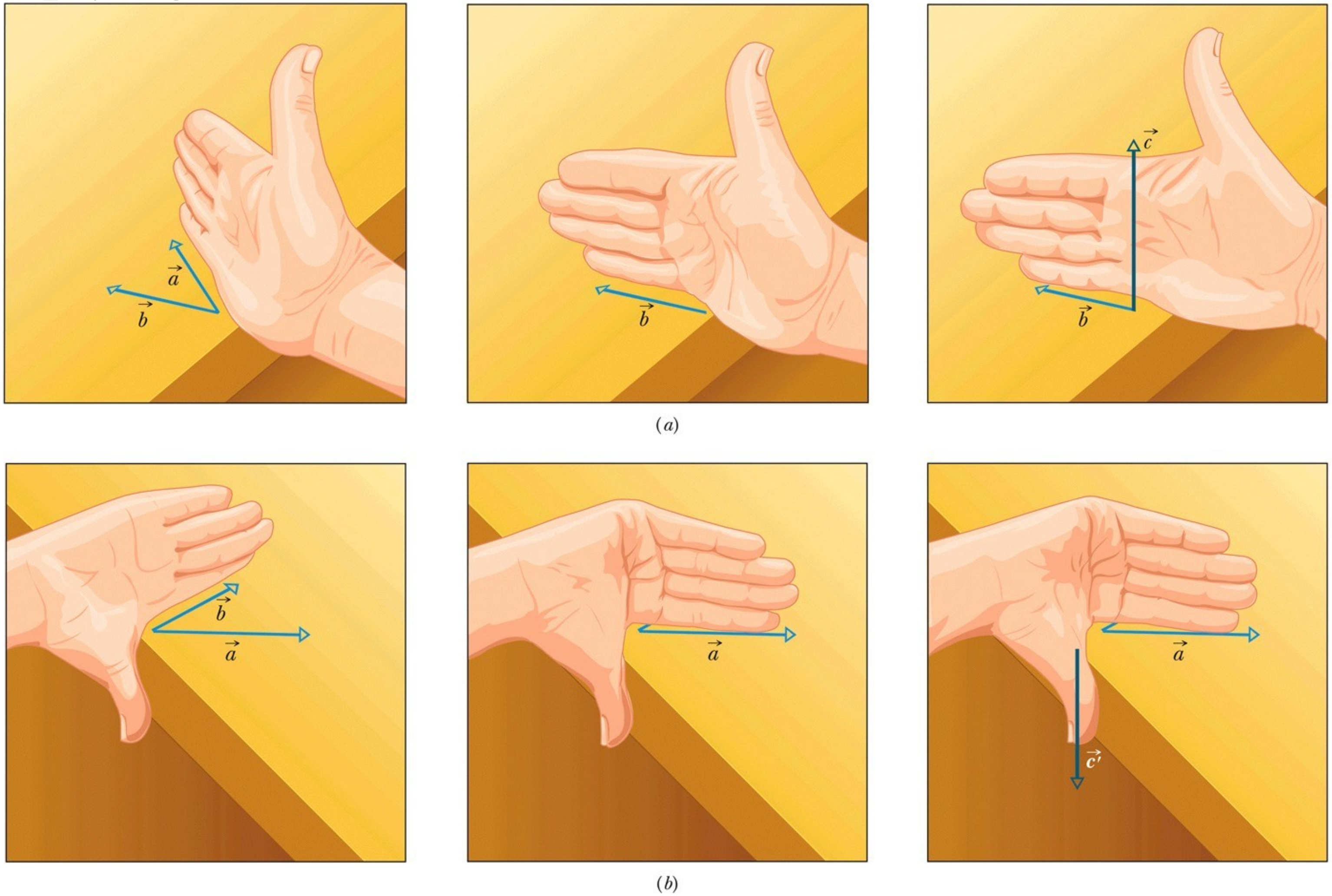
C. Multiplying a vector with a vector: Vector (Cross) Product

$\vec{a} \times \vec{b}$

$\vec{b} \times \vec{a}$

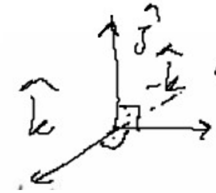
$\downarrow \vec{c}$

lengths are same but directions are not



The upper shows vector ***a*** cross vector ***b***, the lower shows vector ***b*** cross vector ***a***.

C. Multiplying a vector with a vector: Vector (Cross) Product



RH cartesian

Components of cross product

$$\left.\begin{aligned}\vec{a}&=a_x\hat{i}+a_y\hat{j}+a_z\hat{k}\\ \vec{b}&=b_x\hat{i}+b_y\hat{j}+b_z\hat{k}\end{aligned}\right\}\ \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ a_x&a_y&a_z\\ b_x&b_y&b_z\end{vmatrix}\sim \boxed{\text{deteminant}}\ \text{1st way}$$

$$\vec{a}\times\vec{b}=\boxed{\begin{aligned}&\left(\cancel{a_xb_x\,\hat{i}\times\hat{i}}+a_xb_y\,\hat{i}\times\hat{j}+a_xb_z\,\hat{i}\times\hat{k}\right)+\\ &\left(a_yb_x\,\hat{j}\times\hat{i}+\cancel{a_yb_y\,\hat{j}\times\hat{j}}+a_yb_z\,\hat{j}\times\hat{k}\right)+\\ &\left(a_zb_x\,\hat{k}\times\hat{i}+a_zb_y\,\hat{k}\times\hat{j}+\cancel{a_zb_z\,\hat{k}\times\hat{k}}\right)\end{aligned}}$$

$\hat{i}\rightarrow\hat{j}\rightarrow\hat{k}$ 2nd way

otherwise minus sign

$\hat{i}\times\hat{j}=\hat{k}$

$\hat{j}\times\hat{i}=-\hat{k}$

$$=a_xb_y\,\hat{k}+a_xb_z(-\hat{j})+a_yb_x(-\hat{k})+a_yb_z\,\hat{i}+a_zb_x\,\hat{j}+a_zb_y(-\hat{i})$$

$$\boxed{\vec{a}\times\vec{b}=(a_yb_z-a_zb_y)\,\hat{i}+(a_zb_x-a_xb_z)\,\hat{j}+(a_xb_y-a_yb_x)\,\hat{k}}$$

Components

$$\vec{c}=\vec{a}\times\vec{b}$$
$$=c_x\hat{i}+c_y\hat{j}+c_z\hat{k}$$

$$\boxed{\begin{aligned}&\hat{i}\times\hat{i}\sim|\hat{i}|\,|\hat{i}|\,\underbrace{\sin\varphi}_{0}\sim 0\\ &\hat{i}\times\hat{j}\sim\underbrace{|\hat{i}|}_{1}\,\underbrace{|\hat{j}|}_{1}\,\underbrace{\sin 90}_{1}\sim 1\,\hat{k}\end{aligned}}$$

3-8 Multiplying Vectors

- The cross product is not commutative

order IS important!

$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b}).$$

indicates the direction

- To evaluate, we distribute over components:

$$\vec{a} \times \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \times (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$

$$a_x\hat{\mathrm{i}} \times b_x\hat{\mathrm{i}} = a_x b_x(\hat{\mathrm{i}} \times \hat{\mathrm{i}}) = 0,$$

$$a_x\hat{\mathrm{i}} \times b_y\hat{\mathrm{j}} = a_x b_y(\hat{\mathrm{i}} \times \hat{\mathrm{j}}) = a_x b_y\hat{\mathrm{k}}.$$

- Therefore, by expanding

$$\vec{a} \times \vec{b} = (a_y b_z - b_y a_z)\hat{\mathrm{i}} + (a_z b_x - b_z a_x)\hat{\mathrm{j}} + (a_x b_y - b_x a_y)\hat{\mathrm{k}}.$$

nothing but 2nd way

Example:

In Fig. 3-20, vector $\vec{a}$ lies in the xy plane, has a magnitude of 18 units and points in a direction 250° from the positive direction of the x axis. Also, vector $\vec{b}$ has a magnitude of 12 units and points in the positive direction of the z axis. What is the vector product $\vec{c} = \vec{a} \times \vec{b}$?

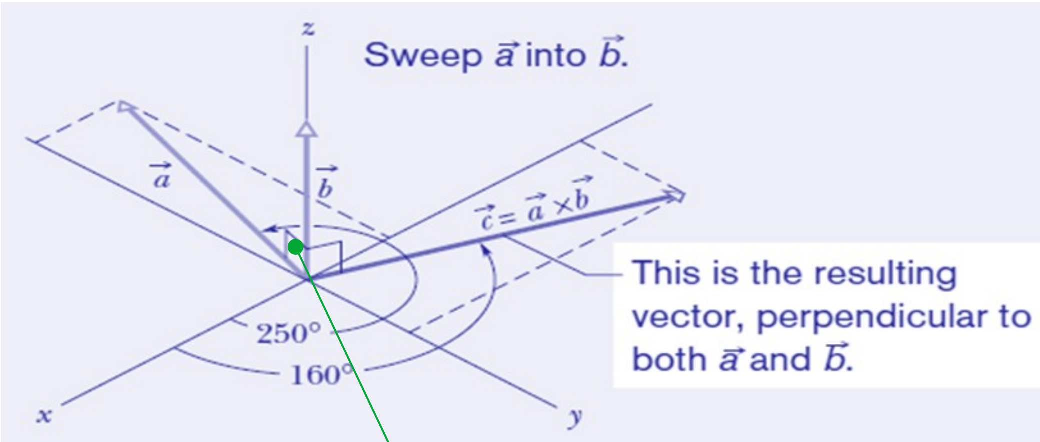


Fig. 3-20 Vector $\vec{c}$ (in the xy plane) is the vector (or cross) product of vectors $\vec{a}$ and $\vec{b}$.

KEY IDEA

When we have two vectors in magnitude-angle notation, we find the magnitude of their cross product with Eq. 3-27 and the direction of their cross product with the right-hand rule of Fig. 3-19.

$$\vec{c} = \vec{a} \times \vec{b} \Rightarrow \underbrace{|\vec{a}|}_{18}\,\underbrace{|\vec{b}|}_{12}\,\sin\underbrace{\phi}_{90^\circ}$$

$|\vec{c}| = 216$ & angle w.r.t. $+x$ axis $\Rightarrow 160^\circ$

Calculations: For the magnitude we write

$$c = ab \sin\phi = (18)(12)(\sin 90^\circ) = 216. \qquad \text{(Answer)}$$

To determine the direction in Fig. 3-20, imagine placing the fingers of your right hand around a line perpendicular to the plane of $\vec{a}$ and $\vec{b}$ (the line on which $\vec{c}$ is shown) such that your fingers sweep $\vec{a}$ into $\vec{b}$. Your outstretched thumb then gives the direction of $\vec{c}$. Thus, as shown in the figure, $\vec{c}$ lies in the xy plane. Because its direction is perpendicular to the direction of $\vec{a}$ (a cross product always gives a perpendicular vector), it is at an angle of

$$250^\circ - 90^\circ = 160^\circ \qquad \text{(Answer)}$$

from the positive direction of the x axis.

İZMİR KÂTİP ÇELEBİ ÜNİVERSİTESİ

Example :

If $\vec{a} = 3\hat{\imath} - 4\hat{\jmath}$ and $\vec{b} = -2\hat{\imath} + 3\hat{k}$, what is $\vec{c} = \vec{a} \times \vec{b}$?

$x \quad y \quad x \quad z$

KEY IDEA

When two vectors are in unit-vector notation, we can find their cross product by using the distributive law.

Calculations: Here we write

$$\vec{c} = (3\hat{\imath} - 4\hat{\jmath}) \times (-2\hat{\imath} + 3\hat{k})$$
$$= \cancel{3\hat{\imath} \times (-2\hat{\imath})} + 3\hat{\imath} \times 3\hat{k} + (-4\hat{\jmath}) \times (-2\hat{\imath}) + (-4\hat{\jmath}) \times 3\hat{k}.$$

$\hat{\imath} \to \hat{\jmath} \to \hat{k} \to \hat{\imath}$

$\hat{\imath} \times \hat{k} \to -\hat{\jmath}$
$\hat{\jmath} \times \hat{\imath} \to -\hat{k}$
$\hat{\jmath} \times \hat{k} \to +\hat{\imath}$

$$(a_y b_z - a_z b_y)\,\hat{\imath} + (a_z b_x - a_x b_z)\,\hat{\jmath} + (a_x b_y - a_y b_x)\,\hat{k}$$
$$(-4\cdot 3 - 0\cdot 0)\,\hat{\imath} + (0(-2) - 3\cdot 3)\,\hat{\jmath} + (3\cdot 0 - (-4)(-2))\,\hat{k}$$
$$\vec{c} = -12\hat{\imath} - 9\hat{\jmath} - 8\hat{k}$$

We next evaluate each term with Eq. 3-27, finding the direction with the right-hand rule. For the first term here, the angle ϕ between the two vectors being crossed is 0. For the other terms, ϕ is 90°. We find

$$\vec{c} = -6(0) + 9(-\hat{\jmath}) + 8(-\hat{k}) - 12\hat{\imath}$$
$$= -12\hat{\imath} - 9\hat{\jmath} - 8\hat{k}. \qquad \text{(Answer)}$$

This vector $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, a fact you can check by showing that $\vec{c}\cdot\vec{a} = 0$ and $\vec{c}\cdot\vec{b} = 0$; that is, there is no component of $\vec{c}$ along the direction of either $\vec{a}$ or $\vec{b}$.

Example:

no vector algebra
but component wise ops: { x-components   y   z

Example:

$\vec{a} = 4.2\hat{i} - 1.5\hat{j}$

$\vec{b} = -1.6\hat{i} + 2.9\hat{j}$

$\vec{c} = -3.7\hat{j}$

$\vec{r} = \vec{a}+\vec{b}+\vec{c}$

magnitude & angle

$\vec{r} = (4.2-1.6)\hat{i} + (-1.5+2.9-3.7)\hat{j} + (0+0+0)\hat{k}$

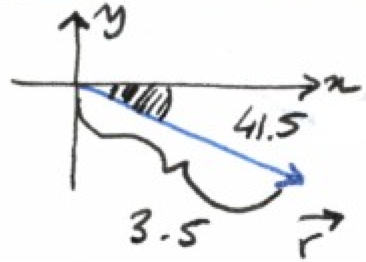
$\boxed{\vec{r} = 2.6\hat{i} + (-2.3)\hat{j}}$

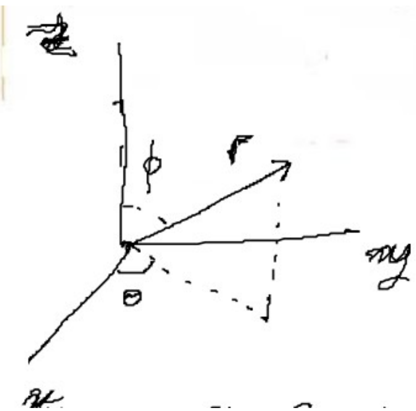
$|\vec{r}| = \sqrt{(2.6)^2+(-2.3)^2} \simeq 3.5$

$\theta = \tan^{-1}\frac{-2.3}{2.6} = -41.5^\circ$

cw

magnitude-angle notation





$x, y, z \to r, \theta, \phi$

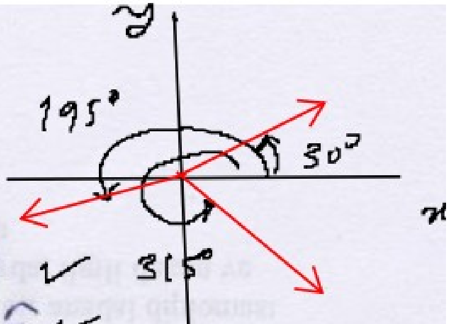
cartesian coordinate system   polar coordinates

FYI

1. Three vectors **a**, **b**, and **c** each have a magnitude of 50 m and lie in an xy-plane. Their directions relative to the positive direction of the axis are 30°, 195°, and 315°, respectively. What are (a) the magnitude and (b) the angle of the vector **a** + **b** + **c**, and (c) the magnitude and (d) the angle of **a - b** + **c**? What are the (e) magnitude and (f) angle of a fourth vector such that (**a** + **b**) – (**c** + **d**) =0?

17) $|\vec{a}| = |\vec{b}| = |\vec{c}| = 50\ m$ & in xy-plane

30°   195°   315° wrt +x-axis



i) $\vec{a}+\vec{b}+\vec{c} = ?$ in magnitude and direction

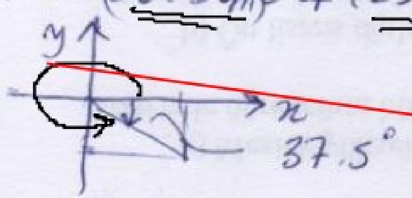
$\vec{a} = |\vec{a}| \cos 30° \,\hat{i} + |\vec{a}| \sin 30° \,\hat{j} = 43.30\,\hat{i} + 25\,\hat{j}$

$\vec{b} = (50m) \cos 195° \,\hat{i} + (50m) \sin 195° \,\hat{j} = -48.30\,\hat{i} - 12.94\,\hat{j}$

$\vec{c} = (50m) \cos 315° \,\hat{i} + (50m) \sin 315° \,\hat{j} = 35.36\,\hat{i} - 35.36\,\hat{j}$

$\vec{a}+\vec{b}+\vec{c} = (50m)(\cos 30° + \cos 195° + \cos 315°)\,\hat{i} + (50m)(\sin 30° + \sin 195° + \sin 315°)\,\hat{j}$

$\vec{r} = (30.36m)\,\hat{i} + (23.30m)(-\hat{j}) \rightarrow$ magnitude $|\vec{r}| = \sqrt{(30.36m)^2 + (23.30m)^2} = 38.27m$



angle $\theta = \tan^{-1} \dfrac{-23.30}{30.36} = -37.5°$ cw

$360 + (-37.5) = +322.5°$ ccw

ii) $\vec{a} - \vec{b} + \vec{c} = ?$

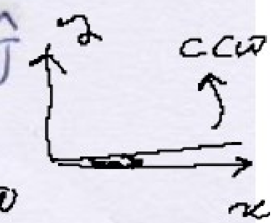
1. (Continued) Three vectors **a**, **b**, and **c** each have a magnitude of 50 m and lie in an xy-plane. Their directions relative to the positive direction of the axis are 30°, 195°, and 315°, respectively. What are (a) the magnitude and (b) the angle of the vector **a** + **b** + **c**, and (c) the magnitude and (d) the angle of **a** - **b** + **c**? What are the (e) magnitude and (f) angle of a fourth vector such that (**a** + **b**) – (**c** + **d**) =0?

ii) $\vec{a} - \vec{b} + \vec{c} = ?$ $\quad 360 + (-37.5) = +322.5$ ccw

$= (43.30 - (-48.30) + 35.36)_m \hat{i} + (25 - (-12.94) - 35.36)_m \hat{j}$

$\vec{r} = 126.95_m \hat{i} + 2.59_m \hat{j} \rightarrow$ magnitude $|\vec{r}| = 126.95$ m

angle $\theta = +1.17°$ → ccw

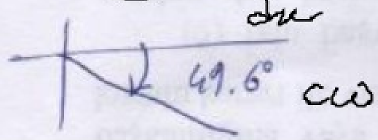


iii) a fourth vector such that $(\vec{a} + \vec{b}) - (\vec{c} + \vec{d}) = 0$

$\rightarrow \underline{\underline{\vec{d}}} = (\vec{a} + \vec{b} - \vec{c}) = (43.30 + (-48.30) - 35.36)\hat{i} + (25 + (-12.94) - (-35.36))\hat{j}$

$= \underbrace{-40.35}_{d_x}\hat{i} + \underbrace{47.42}_{d_y}\hat{j} \rightarrow$ magnitude $|\vec{d}| = 62.26$ m

angle $\underline{\underline{\theta = -49.6°}}$



3 Solved Problems

2. Three vectors are given by **a** = 3.0**i** + 3.0**j** - 2.0**k**, **b** = -1.0**i** - 4.0**j** + 2.0**k**, and **c** = 2.0**i** + 2.0**j** + 1.0**k**. Find (a) **a . (b x c)**, (b) **a . (b + c)**, and (c) **a x (b + c)**.

scalar scalar vector

37) $\vec{a} = 3.0\,\hat{i} + 3.0\,\hat{j} - 2.0\,\hat{k}$

$\vec{b} = -1.0\,\hat{i} - 4.0\,\hat{j} + 2.0\,\hat{k}$

$c = 2.0\,\hat{i} + 2.0\,\hat{j} + 1.0\,\hat{k}$

i) $\vec{a}\cdot(\vec{b}\times\vec{c}) = ?$

first

$$\vec{b}\times\vec{c} = (b_y c_z - b_z c_y)\,\hat{i} + (b_z c_x - b_x c_z)\,\hat{j} + (b_x c_y - c_x b_y)\,\hat{k}$$

$$= (-4\times1 - 2\times2)\,\hat{i} + (2\times2 - (-1)\times1)\,\hat{j} + (-1\times2 - 2(-4))\,\hat{k}$$

$$= -8\,\hat{i} + 5\,\hat{j} + 6\,\hat{k}$$ now $\vec{a}$ in dot product

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (3\times-8\,\hat{i}\cdot\hat{i} + 3\times5\,\hat{j}\cdot\hat{j} - 2\times6\,\hat{k}\cdot\hat{k}) = \underline{-21}$$

ii) $\vec{a}\cdot(\vec{b}+\vec{c}) = ?$ $(\vec{b}+\vec{c}) = 1.0\,\hat{i} - 2.0\,\hat{j} + 3.0\,\hat{k}$

$$\vec{a}\cdot(\vec{b}+\vec{c}) = 1.0\times3.0 - 2.0\times3.0 + 3.0\times2.0 = \underline{-9.0}$$

iii) $\vec{a}\times(\vec{b}+\vec{c}) = ?$

$$\vec{a}\times(\vec{b}+\vec{c}) = (3.0\times3 - (-2)(-2))\,\hat{i} + ((-2)1 - 3(3))\,\hat{j} + (3(-2) - 3\times1)\,\hat{k}$$

$$= 5\,\hat{i} - 11\,\hat{j} - 9\,\hat{k}$$

i) $\vec{a} = 3\hat{i} + 3\hat{j} - 2\hat{k}$

$\vec{b} = -1\hat{i} - 4\hat{j} + 2\hat{k}$

$\vec{c} = 2\hat{i} + 2\hat{j} + 1\hat{k}$

$\vec{a} \cdot \underbrace{(\vec{b} \times \vec{c})}_{\text{vector}}$ cross product

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -4 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$

$$\underbrace{(-4 \cdot 1 - 2 \cdot 2)}_{b_y c_z - b_z c_y}\hat{i} + \underbrace{(2 \cdot 2 - (-1)1)}_{b_z c_x - b_x c_z}\hat{j} + \underbrace{((-1)2 - (-4)2)}_{b_x c_y - b_y c_x}\hat{k}$$

$-8\hat{i} \qquad 5\hat{j} \qquad 6\hat{k}$ a vector

$(3\hat{i} + 3\hat{j} - 2\hat{k}) \cdot (-8\hat{i} + 5\hat{j} + 6\hat{k})$

$3(-8) + 3 \cdot 5 + (-2)(6) = -21$ ✓

$\cdot \sim \cos\theta$

$\times \sim \sin\theta$

zero terms $\hat{i} \cdot \hat{j} \sim \underbrace{|\hat{i}|}_{1}\underbrace{|\hat{j}|}_{1}\underbrace{\cos 90°}_{0}$

$\hat{i} \cdot \hat{i}, \hat{j} \cdot \hat{j}, \hat{k} \cdot \hat{k}$

$\hat{i} \cdot \hat{i} = |\hat{i}||\hat{i}|\underbrace{\cos 0}_{1}$

non-zero terms $\underbrace{\qquad\qquad}_{1}$

ii) $\vec{b} + \vec{c} = 1\hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{a} \times (\vec{b} + \vec{c})$

$$\begin{array}{c|ccc|} & \hat{i} & \hat{j} & \hat{k} \\ \vec{a} & 3 & 3 & -2 \\ \vec{d} & 1 & -2 & 3 \end{array}$$

ijk order ← interchange the indices

$$\rightsquigarrow (a_y d_z - a_z d_y)\hat{i} + (a_z d_x - a_x d_z)\hat{j} + (a_x d_y - a_y d_x)\hat{k} =$$

$$\underbrace{(3 \cdot 3 - (-2)(-2))}_{5}\hat{i} + \underbrace{((-2)(1) - 3 \cdot 3)}_{-11}\hat{j} + \underbrace{(3(-2) - 3 \cdot 1)}_{-9}\hat{k}$$ ✓

3. Vectors **A** and **B** lie in an xy-plane. **A** has magnitude 8.0 and an angle 130º; **B** has components $B_x = -7.72$ and $B_y = -9.20$. a) What are 5**A**.**B** and 4**A**x3**B** in unit vector notation? b) What is (3î + 5ĵ)x (4**A**x3**B**)? Find magnitude and angle of resultant vector.

i) $\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$, $\boxed{\begin{array}{l} A_x = A\cdot\cos 130 = -5.14 \\ A_y = A\cdot\sin 130 = 6.13 \end{array}}$, $B_x = -7.72$, $B_y = -9.20$

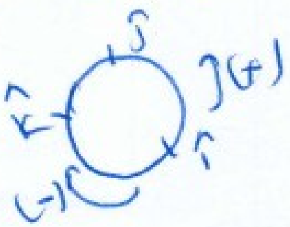
$5\,\vec{A}\cdot\vec{B} = 5\left[(-5.14)(-7.72) + (6.13)(-9.20)\right]$

$= 5[39.68 - 56.40] = \boxed{83.58}$

$$4\vec{A}\times 3\vec{B} = \begin{vmatrix} \overset{(+)}{\hat{i}} & \overset{(-)}{\hat{j}} & \overset{(+)}{\hat{k}} \\ -20.56 & 24.52 & 0 \\ -23.16 & -27.6 & 0 \end{vmatrix} = \left[(-20.56)(-27.6) - (-23.16)(24.52)\right]\hat{k}$$

$= [567.5 + 567.9]\hat{k} = \boxed{1135.4\,\hat{k}}$

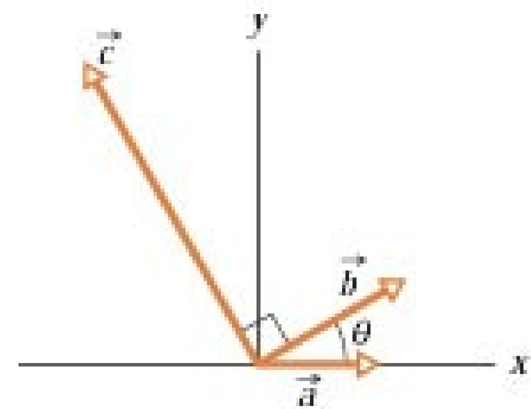
ii) $(3\hat{i} + 5\hat{j}) \times [1135.4\,\hat{k}] = \boxed{3406.2(-\hat{j}) + (5677)\hat{i}} = \vec{F} = F_x\hat{i} + F_y\hat{j}$



Magnitude $= \sqrt{(5677)^2 + (-3406.2)^2} = \boxed{6620.5}$

$\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{-3406.2}{5677}\right) = \boxed{-31^\circ}$ or $\boxed{329^\circ}$

4. The three vectors in Figure have magnitudes a=3.00 m, b=4.00 m, and c=10.0 m and angle ϴ=30.0°. What are (a) the *x* component and (b) the *y* component of **a**; (c) the *x* component and (d) the *y* component of **b**; and (e) the *x* component and (f) the *y* component of **c**? If **c** = p**a** + q**b**, what are the values of (g) p and (h) q?



43) $|\vec{a}| = 3.00$ m, $|\vec{b}| = 4.00$ m, $|\vec{c}| = 10.0$ m & $\theta = 30$

$a_x = 3.00 \cos 0 = 3.00$ m ; $b_x = 4.00 \cos 30° = 3.46$ m ; $c_x = 10.0 \cos 120° = -5.00$ m

$a_y = 3.00 \sin 0 = 0.00$; $b_y = 4.00 \sin 30 = 2.00$ m ; $c_y = 10.0 \sin 120° = 8.66$ m

$\vec{c} = p\vec{a} + q\vec{b}$ what p & q?

$-5.00\,m\,\hat{i} + 8.66\,m\,\hat{j} = p\,3.00\,m\,\hat{i} + q\,3.46\,m\,\hat{i} + q\,2.00\,m\,\hat{j}$

$\cancel{p = \frac{5.00}{3.00}} = q = \frac{8.66\,m}{2.00\,m} = 4.33$; $3p + 3.46q = -5$

$p = \frac{-5 - 3.46 \times 4.33}{3} = -6.66$

i.e. ii $a_x = |\vec{a}| \cos\theta$ & $a_y = |\vec{a}| \sin\theta$

$\theta = 0°$ $= 3 \cos 0 = 3\,m$ $3 \sin 0 = 0$

iii $\theta = 30°$ $b_x = \cancel{4}\, \cos 30 = 3.46\,m$ $\Big\{ b_y = \cancel{4}\, \sin 30 = 2\,m$

iv & v $\theta = 120°$ $c_x = 10 \cos 120 = \underline{\underline{-5\,m}}$ $\Big\} c_y = 10 \sin 120 = 8.66\,m$

vi) $\vec{c} = p\overbrace{\vec{a}}^{} + q\overbrace{\vec{b}}^{}$

$-5\,m\,\hat{i} + 8.66\,m\,\hat{j} = p(3\,m\,\hat{i} + 0\,\hat{j}) + q(3.46\,\hat{i} + 2\,\hat{j})$

$-5\,m\,\hat{i} + 8.66\,m\,\hat{j} = (3p + 3.46\,q)\,\hat{i} + 2q\,\hat{j}$

2 unknowns $\rightsquigarrow$ 2 equations

$-5 = 3p + 3.46\,q$ from $\hat{i}$ component

$8.66 = 2q$ from $\hat{j}$ component

$q = 4.33$ //

$p = \dfrac{-5 - 3.46 \times 4.33}{3} = -\underline{6.66}$

5. Given two vectors, **A**= 5i − 6.5j and **B**= −3.5i +7j. A third vector lies in the xy-plane. Vector **C** is perpendicular to vector **A**, and the scalar product of **C** with **B** is 15.0. From this information, find the components of vector **C**.

$\vec{A}$ and $\vec{C}$ are perpendicular, so $\vec{A}\cdot\vec{C} = 0$

$$A_x C_x + A_y C_y = 0$$

$$5.0\,C_x - 6.5\,C_y = 0 \qquad (1)$$

$\vec{B}\cdot\vec{C} = 15.0$, so $-3.5\,C_x + 7.0\,C_y = 15.0 \qquad (2)$

We have two equations in two unknowns C_x any C_y.
Solving gives $C_x = 8.0$ and $C_y = 6.1$

Scalars and Vectors

- Scalars have magnitude only
- Vectors have magnitude and direction
- Both have units!

Vector Components

- Given by **Eq. (3-5)**

$$a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta,$$

- Related back by **Eq. (3-6)**

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\theta = \frac{a_y}{a_x}$$

Adding Geometrically

- Obeys commutative and associative laws

Eq. (3-2)

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}).$$

Eq. (3-3)

Unit Vector Notation

- We can write vectors in terms of unit

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}},$$ **Eq. (3-7)**

Adding by Components

- Add component-by-component

$$r_x = a_x + b_x$$
$$r_y = a_y + b_y$$
$$r_z = a_z + b_z.$$

Eqs. (3-10) - (3-12)

Scalar Product

- Dot product $\vec{a}\cdot\vec{b} = ab\cos\phi,$ **Eq. (3-20)**

$$\vec{a}\cdot\vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}})\cdot(b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$ **Eq. (3-22)**

Scalar Times a Vector

- Product is a new vector. Magnitude is multiplied by scalar. Direction is same or opposite.

Cross Product

- Produces a new vector in perpendicular direction
- Direction determined by right-hand rule

$$c = ab\sin\phi,$$ **Eq. (3-24)**