Chapter 10 - Rotation

Motion of translation → along a straight line

Motion of rotation → a rigid body turns around an axis (about COM!) SLN

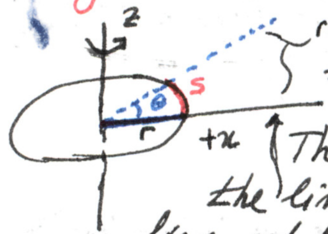
- → rotational acceleration (constant or not!)
- → torque (instead of force)
- → inertia (instead of mass)

- Rotational Variables

A rigid body about a fixed axis SLN Fig. 10-2

1) Angular Position

$\theta(t)$ time dependence



reference line, perpendicular to rotation axis

The angular position of the line is the angle of the line relative to a fixed direction.

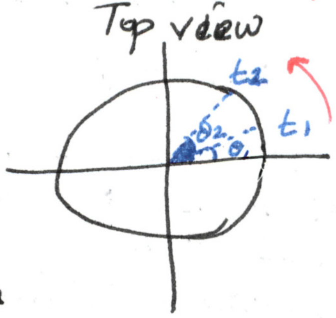
$\boxed{\theta = \frac{s}{r}}$ — circular arc length / radius of circle

zero angular position

- radian measure
- angle

$360° = 2\pi \text{ rad} \equiv 1 \text{revolution}$

2) Angular Displacement $\Delta\theta$



$\boxed{\Delta\theta = \theta_2 - \theta_1}$

CCW → (+)

CW → (−)

3) Angular Velocity

ω (rad/s or rev/s)

$$\boxed{\underset{\text{average}}{\omega_{avg}} = \frac{\theta_2-\theta_1}{t_2-t_1} = \frac{\Delta\theta}{\Delta t}} \quad \boxed{\underset{\text{instantaneous}}{\omega} = \lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}}$$

- CCW → (+)
- CW → (−)

- magnitude is called angular speed, ω

→ $\theta = \int \omega dt$ $\quad \omega = \int \alpha dt$

4) Angular Acceleration

α (rad/s² or rev/s²)

$$\boxed{\alpha_{avg} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2-\omega_1}{\Delta t}} \quad \boxed{\alpha = \lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}}$$

Example Angular velocity derived from angular position

Disk is rotating as $\theta(t) = -1.00 - 0.600t + 0.250t^2$ SLN Fig. 10-5a

i) Angular position of reference line at $t = -2.0$ s, 0 s, 4 s, $\theta = 0$ points

$t = -2 \rightarrow \theta(t=-2) = -1 - 0.6(-2) + 0.25(-2)^2 = 1.2 \text{ rad} \Rightarrow \left.\begin{matrix} 2\pi \text{ rad} & 360° \\ 1.2 \text{ rad} & x \end{matrix}\right\} \theta = 69°$ ccw

SLN Fig. 10-5b for the rest $\left\{\begin{matrix} t=0 \rightarrow \theta = -1.00 \text{ rad} \rightarrow -57° \text{ cw} \\ t=4 \rightarrow \theta = 0.60 \text{ rad} \rightarrow 34° \text{ ccw} \end{matrix}\right.$

$\left.\right\}$ θ=0 points. reference line is aligned with zero angular position (s2, s4)

ii) $t_{min} = ? \xrightarrow{\text{that makes}} \theta(t)$ minimum.

SLN. Fig. 10-5c → What about angular acceleration!

To have a minimum $\left.\frac{d\theta}{dt}\right|_{t=t_{min}} = 0 \rightarrow -0.6 + 0.5t = 0 \rightarrow t = 1.20\text{s}$ (see Fig. 10-5b)

$\theta(t=1.20s) = -1.36 \text{rad} \sim -77.9°$ maximum cw rotation!

iii) $\left.\begin{matrix} t=0 \rightarrow \omega(0) = -0.6 \text{ rad/s} \\ t=1 \rightarrow \omega(1) = -0.1 \quad '' \\ t=2 \rightarrow \omega(2) = 0.4 \quad '' \end{matrix}\right\} \frac{d\theta}{dt} = \omega = -0.6 + 0.5t$

Example Anguler velocity derived from angular acceleration

$\alpha = 5t^3 - 4t$

$t=0 \begin{cases} \omega = 5 \text{ rad/s} \\ \theta = 2 \text{ rad} \end{cases}$

i) $\omega(t) = ?$ $\int d\omega = \int \alpha dt \rightarrow \omega = \int (5t^3 - 4t)dt = \frac{5}{4}t^4 - \frac{4}{2}t^2 + C$

$\omega(t=0) = 5 = \frac{5}{4}0^4 - \frac{4}{2}0^2 + C \Rightarrow \omega(t) = \frac{5}{4}t^4 - 2t^2 + 5$

ii) $\theta(t) = ?$ $\int d\theta = \int \omega dt \rightarrow \theta = \int (\frac{5}{4}t^4 - 2t^2 + 5)dt = \frac{1}{4}t^5 - \frac{2}{3}t^2 + 5t + C$

$\theta(t=0) = 2 \rightarrow \theta(t) = \frac{t^5}{4} - \frac{2}{3}t^2 + t + 2$

- Are Anguler Quantities vectors?

Angular Displacement, $\Delta\theta$ → Can not be treated as vectors. Does not obey to vector arithmetics.

Angular Velocity, ω } Can be treated as vectors } SLN Fig. 10-6 Fig. 10-7 Directions of vector and motion are differnt } ω and α can be represented by ± sign. ccw (+) cw (−)

Anguler Acceleration, α

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SLN Rotation with Constant Angular Acceleration Table 10-1

Example Constant angular acceleration, gridstone

$\alpha = 0.35 \text{ rad/s}^2$

$\omega_0 = -4.6 \text{ rad/s}$

$\theta_0 = 0$ (refrence line)

SLN Fig. 10-8

i) $t = ?$ at $\theta = 5 \text{ rev}$ $\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$ $5 \times 2\pi \text{ rad} = -4.6 \text{ rad/s}\, t + 0.35 \text{ rad/s}^2\, t^2$

$\theta - \theta_0$

$\Rightarrow t = 32 \text{ s}$

ii) $\alpha \rightarrow$ positive, $\omega_0 \rightarrow$ negative } initially slows down, momentarily stops, rotates again

cw ccw since α (+) θ (+)

iii) $t = ?$ at $\omega = 0$ $\overset{0}{\cancel{\omega}} = \omega_0 + \alpha t \rightarrow +4.6 \text{ rad/s} = 0.35 \text{ rad/s}^2\, t$

$\Rightarrow t = 13 \text{ s}$

Example Constant angular acceleration, riding a Rotor

$\omega_0 = 3.4 \text{ rad/s}$

$\omega = 2.0 \text{ rad/s}$

$\theta - \theta_0 = 20.0 \text{ rev} \left(\times 2\pi \frac{\text{rad}}{\text{rev}}\right)$

constant angular acceleration

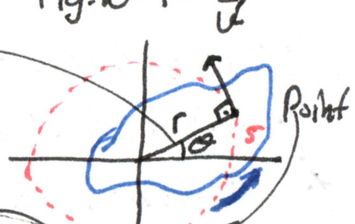
i) $\alpha = ?$ $\omega = \omega_0 + \alpha t$, $\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$ } $\theta - \theta_0 = \omega_0\left(\frac{\omega - \omega_0}{\alpha}\right) + \frac{1}{2}\alpha\left(\frac{\omega - \omega_0}{\alpha}\right)^2 \Rightarrow \alpha = -0.0301 \text{ rad}$

slowing down

ii) $t = \frac{\omega - \omega_0}{\alpha} = \frac{2.0 \text{ rad/s} - 3.4 \text{ rad/s}}{-0.0301 \text{ rad/s}^2} = 46.5 \text{ s}$

Relating the Linear and Angular Variables

Fig. 10-9a



s, v, a } can be related to angular counterparts } θ, ω, α by r: the perpendicular distance of the point from the rotation axis

Point P makes a rotation. velocity v, distance s

Object makes a rotation about a fixed axis. ω

} $s = \theta r$, $v = \omega r$ (angular speed, linear speed)

⇒ liner speed v depend on the "point"'s location

angular speed ω is same at every "point"

$T = \frac{2\pi r}{v} \rightarrow \boxed{T = \frac{2\pi}{\omega}}$ $2\pi r \leftrightarrow \theta r$: distance travelled

$s = \theta r$

$\frac{ds}{dt} = \frac{d\theta}{dt} r \rightarrow v = \omega r$

$\frac{dv}{dt} = \frac{d\omega}{dt} r \rightarrow a = \alpha r$

SLN Fig. 10-9b

$a \Rightarrow a_t$: tangential component

Remember

$a_r = \frac{v^2}{r} = \omega^2 r$

radially inward (for changes in the direction of linear velocity)

a_t is present when $\alpha \neq 0$

a_r is present when $\omega \neq 0$

Kinetic Enegy of Rotation

$\cancel{KE = \frac{1}{2} M v^2}$

since $v_{com} = 0$

Instead

Suppose that the body is composed of many particles. Then $K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \dots + \frac{1}{2} m_n v_n^2$

$$\boxed{K = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2}$$

$v = \omega r \Rightarrow K = \frac{1}{2} \sum_{i=1}^{n} m_i \omega^2 r_i^2$

same for all particles

$$\Rightarrow \boxed{K = \frac{1}{2} \underbrace{\left(\sum_{i=1}^{n} m_i r_i^2 \right)}_{I} \omega^2 = \frac{1}{2} I \omega^2}$$

I : rotational inertia

Kinetic Enegy of a rigid body in pure rotation

- Tells us how the mass of rotating body is distributed about its axis of rotation.
- It is specified with respect to rotation axis. S.I.N Fig. 10-11
- $kg\,m^2$
- Smaller I means easier rotation. Mass distribution is close to rotation axis.

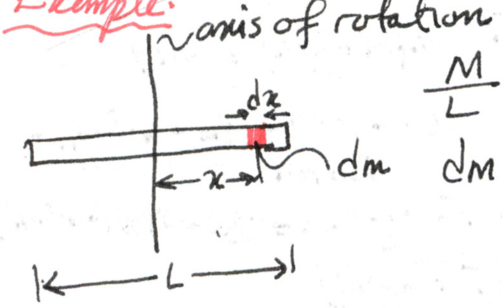
Kinetic Enegy of the body in pure translation $\rightarrow K = \frac{1}{2} M v_{com}^2$

Calculating the Rotational Inetia

A rigid body consists of a few particles $\rightarrow I = \sum m_i r_i^2$ perpendicular distance from rotation axis

of a great many adjacent particles $\rightarrow I = \int r^2 dm$: continuous body

Exemple.



$\frac{M}{L} = \lambda = \frac{dm}{dx}$

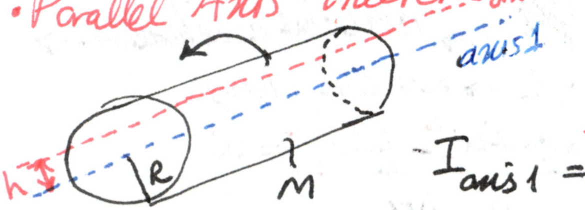
$dm = \lambda dx$
$= \frac{M}{L} dx$

$$I = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx = \frac{M}{L} \int_{-L/2}^{L/2} x^2 dx = \frac{M}{L} \left(\frac{x^3}{3} \right) \Bigg|_{-L/2}^{L/2}$$

$$\boxed{I = \frac{1}{12} M L^2}$$

for thin rod about axis through center perpendicular to length (see Table 10-2e)

• Parallel Axis Theorem



If we know I about an axis (com axis), then I_{com} we can calculate I about an another axis parallel to first one.

$$I_{axis1} = \frac{1}{2} M R^2 = I_{com} \Rightarrow \boxed{I = \frac{1}{2} M R^2 + M h^2}$$

$I_{com} + Mh^2$

Parallel axis theorem

Exomple Rotational Inertia of a two particle system. Fig. 10-13a

i) Rotational axis $\rightarrow$ com axis $I = \sum_{i=1}^{2} m_i r_i^2 = m_1 \left(\frac{L}{2} \right)^2 + m_2 \left(\frac{L}{2} \right)^2 = \frac{ML^2}{4}$, if $M = m_1 + m_2$, $m_1 = m_2$ $\boxed{= \frac{mL^2}{2}}$

ii) Rotational axis $\rightarrow$ at left end $I = I_{com} + Mh^2$

$$\underbrace{= \frac{mL^2}{2} + 2m \left(\frac{L}{2} \right)^2}_{\text{by parallel axis theorem}} \boxed{= mL^2}$$

OR $I = \sum_{i=1}^{2} m_i r_i^2$

$= m_1 (0)^2 + m_2 L^2 \boxed{= mL^2}$

Torque, τ: (To twist)

Does not cause rotation

Resolve $\underbrace{\text{applied force}}_{\vec{F}}$ for rotation into two components $\lessdot$ $\vec{F}_r$: radial component ; F_t : tangential "

Does cause rotation $|\vec{F}|\sin\phi = F_t$

SLN Fig. 10-16 $\tau = rF_t = rF\sin\phi$ → Fig. 10-16b

SI Unit: N.m $\tau = (r\sin\phi)F = r_\perp F$ → Fig. 10-16c

(Be aware that torque is not work! 1J = 1N.m)

· Rotation around an axis → in 1D ⇒ Sign of torque $<$ (+) ccw ; (−) cw

When several forces acting → several torques ⇒ net torque (τ_{net}) is obtained by superposition principle.

Newton's 2nd law for Rotation

SLN Fig. 10-17

Net torque causes an angular acceleration, α. $\boxed{\tau_{net} = I\alpha}$ Newton's 2nd law of rotation

↑ radian measure

Proof: F_t creates a_t $\{ F_t = ma_t \} F_t r = ma_t r \} \tau = m(\alpha r)r \} \tau = (mr^2)\alpha \} \tau = I\alpha$

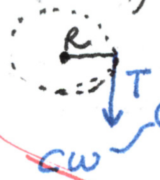
Example. Newton's 2nd Law in Rotational Motion

SLN Fig. 10-18

$M = 2.5\,kg$
$R = 0.2\,m$
$m = 1.2\,kg$
$a = ?$, $\alpha = ?$
$T = ?$

i) $a = ?$ Acceleration of falling block

$T - mg = ma$ ①



$\tau = I\alpha$

$-RT = I\alpha$

$-RT = \frac{1}{2}MR^2(a/r)$

$\Rightarrow T = -\frac{1}{2}Ma$ ②

Combining ① & ②

$-\frac{1}{2}Ma - mg = ma$

$a(m + \frac{1}{2}M) = -mg$

$a = -\frac{2m}{2m+M}g \boxed{= -4.8\,m/s^2}$

ii) $\alpha = ?$ $\alpha = \frac{a}{r} = \frac{-4.8\,m/s^2}{0.20} \boxed{= -24\,rad/s^2}$

iii) $T = -\frac{1}{2}Ma = -\frac{1}{2}(2.5kg)(-4.8m/s^2)$

$\boxed{T = 6.0N}$

Work and Rotational Kinetic Energy

Translational	Rotational
F on a rigid body (m) → acceleration (motion) → does Work → KE can change	τ on a rigid body → rotational acceleration (motion) → does work → KE can change
$\Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = W$	$\Delta K = K_f - K_i = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W$
$W = \int_{x_i}^{x_f} F dx$ { $P = \frac{dW}{dt} = Fv$	$W = \int_{\theta_i}^{\theta_f} \tau d\theta$ } $P = \frac{dW}{dt} = \tau\omega$

KE-Work Theorem

SLN Table 10-3

Example Work, Rotational KE, torque, disk SLN Fig. 10-18

$t = 0 \to \omega = 0$
$T = 6.0N$
$\alpha = -24\,rad/s^2$
KE = ? at $t = 2.5s$
$M = 2.5\,kg$
$R = 0.20\,m$

$KE = \frac{1}{2}I\omega^2$?

I: $\frac{1}{2}MR^2$; $\omega = \cancel{\omega_0} + \alpha t$

$\omega = (-24\,rad/s^2)(2.5s)$

$KE = \frac{1}{4}(2.5kg)(0.20m)^2[(-24\,rad/s^2)(2.5s)]^2$

$\boxed{KE = 90J}$

$W = \tau(\theta_f - \theta_i) = \tau(\cancel{\omega_0 t} + \frac{1}{2}\alpha t^2) = (-TR)(\frac{1}{2}\alpha t^2) = -\frac{1}{2}TR\alpha t^2 = 90J$