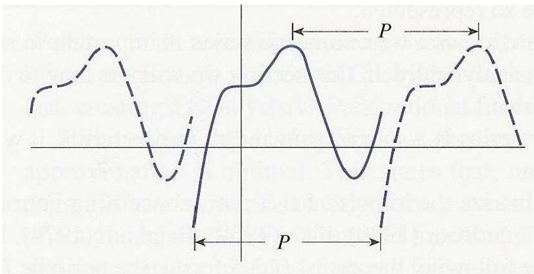

Figure 1: Plot of a periodic function of period P.

1 Fourier Series

- Polynomials are not the on1y functions that can be used to approximate known function
- Another means for representing known functions are approximations that use sines and cosines, called Fourier series
  - Any function can be represented by an infinite sum of sine and cosine terms with the proper coefficients, (possibly with an infinite number of terms)
- Any function, $f(x)$, is periodic of period P if it has the same value for any two x-values, that differ by P, or

$$f(x) = f(x+P) = f(x+2P) = \ldots = f(x-P) = f(x-2P) = \ldots$$

  Figure 1 shows such a periodic function. Observe that the period can be started at any point on the x-axis.
  - $Sin(x)$ and $cos(x)$ are periodic of period 2π
  - $Sin(2x)$ and $cos(2x)$ are periodic of period π
  - $Sin(nx)$ and $cos(nx)$ are periodic of period $2\pi/n$
- We now discuss how to find the As and Bs in a Fourier series of the form

$$f(x) \approx \frac{A_0}{2} + \sum_{n=1}^{\infty}[A_n cos(nx) + B_n sin(nx)] \quad (1)$$

  The determination of the coefficients of a Fourier series (when a given function, $f(x)$, can be so represented) is based on the *property of orthogonality* for sines and cosines. For integer values of n, m:

$$\int_{-\pi}^{\pi} sin(nx) = 0 \tag{2}$$

$$\int_{-\pi}^{\pi} cos(nx) = \begin{cases} 0, & n \neq 0 \\ 2\pi, & n = 0 \end{cases} \tag{3}$$

$$\int_{-\pi}^{\pi} sin(nx)cos(mx) = 0 \tag{4}$$

$$\int_{-\pi}^{\pi} sin(nx)sin(mx) = \begin{cases} 0, & n \neq m \\ \pi, & n = m \end{cases} \tag{5}$$

$$\int_{-\pi}^{\pi} cos(nx)cos(mx) = \begin{cases} 0, & n \neq m \\ \pi, & n = m \end{cases} \tag{6}$$

It is related to the same term used for orthogonal (perpendicular) vectors whose dot product is zero. Many functions, besides sines and cosines, are orthogonal (such as the Chebyshev polynomials).

- To begin, we assume that $f(x)$ is periodic of period 2π and can be represented as in Eq. 1. We find the values of A_n and B_n in Eq. 1 in the following way;
  - Multiply both sides of Eq. 1 by $cos(0x) = 1$, and integrate term by term between the limits of $-\pi$ and π.

    $$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{A_0}{2}dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} A_n cos(nx)dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} B_n sin(nx)$$

    Because of Eqs. 2 and 3, every term on the right vanishes except the first, giving

    $$\int_{-\pi}^{\pi} f(x)dx = \frac{A_0}{2}(2\pi),\ or\ A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

    Hence, A_0 is found and it is equal to twice the average value of $f(x)$ over one period.
  - Multiply both sides of Eq. 1 by $cos(mx)$, where m is any positive integer, and integrate:

    $$\int_{-\pi}^{\pi} cos(mx)f(x)dx = \int_{-\pi}^{\pi} \frac{A_0}{2}cos(mx)dx +$$

    $$\sum_{n=1}^{\infty} \int_{-\pi}^{\pi} A_n cos(mx)cos(nx)dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} B_n cos(mx)sin(nx)dx$$

Because of Eqs. 3,4 and 6 the only nonzero term on the right is when $m = n$ in the first summation, so we get a formula for the As;

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx, \;\; n = 1, 2, 3, \ldots$$

– Multiply both sides of Eq. 1 by $sin(mx)$, where m is any positive integer, and integrate:

$$\int_{-\pi}^{\pi} sin(mx)f(x)dx = \int_{-\pi}^{\pi} \frac{A_0}{2} sin(mx)dx +$$

$$\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} A_n sin(mx)cos(nx)dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} B_n sin(mx)sin(nx)dx$$

Because of Eqs. 2, 4 and 5, the only nonzero term on the right is when $m = n$ in the second summation, so we get a formula for the Bs;

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx, \;\; n = 1, 2, 3, \ldots$$

It is obvious that getting the coefficients of Fourier series involves many integrations.

1.1 Fourier Series for Periods Other Than 2π

- What if the period of $f(x)$ is not 2π? we just make a change of variable. If $f(x)$ is periodic of period P, the function can be considered to have one period between $-P/2$ and $P/2$. The functions $sin(2\pi x/P)$ and $cos(2\pi x/P)$ are periodic between $-P/2$ and $P/2$. The formulae become, for $f(x)$ periodic of period P;

$$A_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)cos\left(\frac{n\pi x}{P/2}\right) dx, \;\; n = 0, 1, 2, \ldots \tag{7}$$

$$B_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)sin\left(\frac{n\pi x}{P/2}\right) dx, \;\; n = 1, 2, 3, \ldots \tag{8}$$

Because a function that is periodic with period P between $-P/2$ and $P/2$ is also periodic with period P between A and $A + P$, the limits of integration in Eqs. 7 and 8 can be from 0 to P.

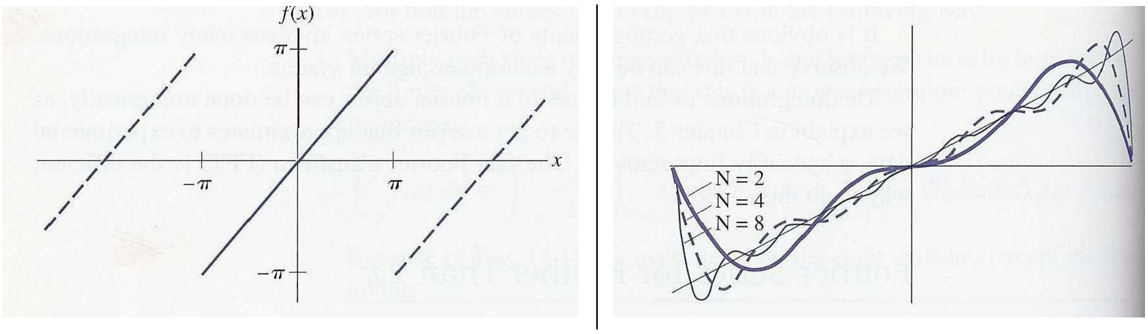

Figure 2: **Left:** Plot of $f(x) = x$, periodic of period 2π,**Right:** Plot of the Fourier series expansion for $N = 2, 4, 8$.

Examples:

1. Let $f(x) = x$ be periodic between $-\pi$ and π. (See Figure 2left). Find the As and Bs of its Fourier expansion. For A_0;

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} xdx = \left[\frac{x^2}{2\pi}\right]_{-\pi}^{\pi} = 0$$

For the other As;

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} xcos(nx)dx = 0$$

For the other Bs;

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} xsin(nx)dx = \frac{2(-1)^{n+1}}{n},\ n = 1, 2, 3, \ldots$$

We then have

$$x \approx 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}sin(nx),\ -\pi < x < \pi$$

Figure 2right shows how the series approximates to the function when only two, four, or eight terms are used.

2. Find the Fourier coefficients for $f(x) = |x|$ on $-\pi$ to π;

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{0} -xdx + \frac{1}{\pi}\int_{0}^{\pi} xdx = \pi$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{0}(-x)cos(nx)dx + \frac{1}{\pi}\int_{0}^{\pi}(x)cos(nx)dx = \begin{cases} 0, & n = 2, 4, 6, \ldots \\ \frac{-4}{(n^2\pi)}, & n = 1, 3, 5, \ldots \end{cases}$$

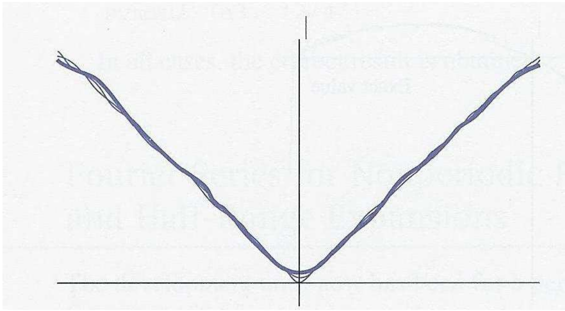

Figure 3: Plot of Fourier series for $|x|$ for $N = 2, 4, 8$.

$$B_n = \frac{1}{\pi}\int_{-\pi}^{0}(-x)sin(nx)dx + \frac{1}{\pi}\int_{0}^{\pi}(x)sin(nx)dx = 0$$

Because the definite integrals are nonzero only for odd values of n, it simplifies to change the index of the summation. The Fourier series is then

$$|x| \approx \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{cos(2n-1)x}{(2n-1)^2}$$

Figure 3 shows how the series approximates the function when two, four, or eight terms are used.

3. Find the Fourier coefficients for $f(x) = x(2-x) = 2x - x^2$ over the interval [-2, 2] if it is periodic of period 4. Equations 7 and 8 apply.

$$A_0 = \frac{2}{4}\int_{-2}^{2}(2x - x^2)dx = \frac{-8}{3}$$

$$A_n = \frac{2}{4}\int_{-2}^{2}(2x - x^2)cos\left(\frac{n\pi x}{2}\right)dx = \frac{16(-1)^{n+1}}{n^2\pi^2},\ n = 1, 2, 3, \ldots$$

$$B_n = \frac{2}{4}\int_{-2}^{2}(2x - x^2)sin\left(\frac{n\pi x}{2}\right)dx = \frac{8(-1)^{n+1}}{n\pi},\ n = 1, 2, 3, \ldots$$

$$x(2-x) \approx \frac{-4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}cos\left(\frac{n\pi x}{2}\right) + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}sin\left(\frac{n\pi x}{2}\right)$$

Figure 4 shows how the series approximates to the function when 40 terms are used.

With MATLAB, two commands are needed because the first result is symbolic and the integration operation does not permit a multiplier (although the 2/4 could be included in the integrand);

```
a3=int('x*(2-x)*cos(3*pi*x/2)',-2,2)
symmul(a3,'2/4')
```

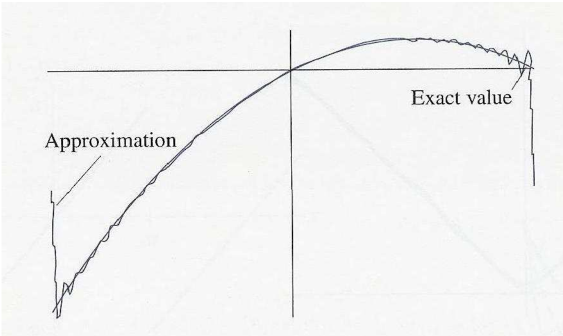

Figure 4: Plot of Fourier series for $x(2-x)$ for $N = 40$.

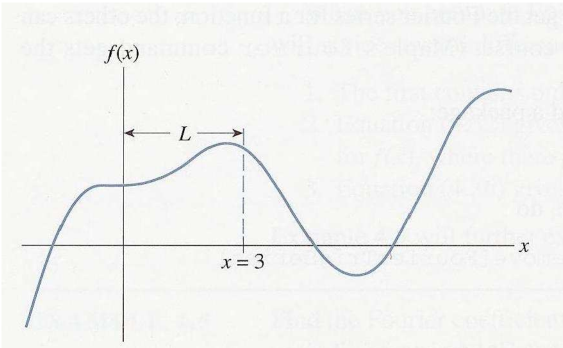

Figure 5: A function, $f(x)$, of interest on [0,3].

1.2 Fourier Series for Nonperiodic Functions and Half-Range Expansions

- The development until now has been for a periodic function. What if $f(x)$ is not periodic? Can we approximate it by a trigonometric series? We assume that we are interested in approximating the function only over a limited interval and we do not care whether the approximation holds outside of that interval.

- Suppose we have a function defined for all x-values, but we are only interested in representing it over (O, L). Figure 5 is typical.

- Because we will ignore the behavior of the function outside of (0, L), we can redefine the behavior outside that interval as we wish Figs. 6left and -right show two possible redefinitions.

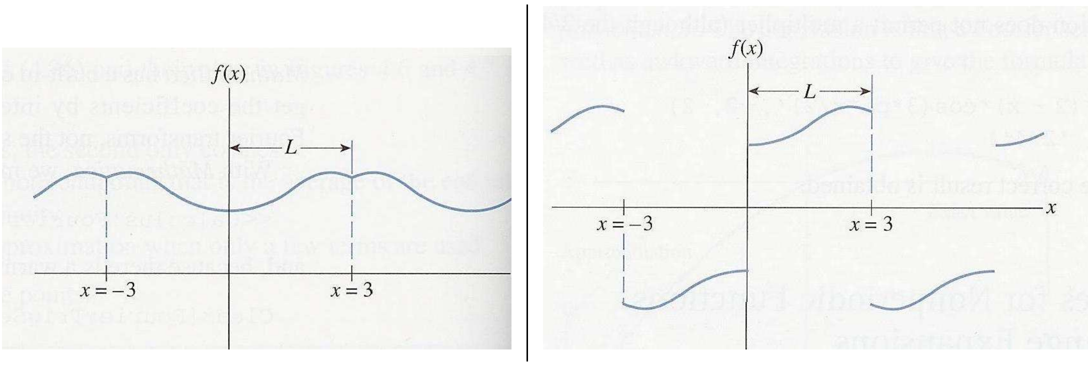

Figure 6: **Left:** Plot of a function reflected about the y-axis, an even function,**Right:** Plot of a function reflected about the origin, an add function.

- In the first redefinition, we have reflected the portion of $f(x)$ about the y-axis and have extended it as a periodic function of period $2L$. This creates an *even* periodic function.

$$f(x)\ is\ even\ if\ f(-x) = f(x)$$

- If we reflect it about the origin and extend it periodically, we create an odd periodic function of period $2L$.

$$f(x)\ is\ odd\ if\ f(-x) = -f(x)$$

It is easy to see that $cos(Cx)$ is an even function and that $sin(Cx)$ is an odd function for any real value of C.

- There are two important relationships for integrals of even and odd functions.

$$if\ f(x)\ is\ even,\ \int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$$

$$if\ f(x)\ is\ odd,\ \int_{-L}^{L} f(x)dx = 0$$

  - the product of two even functions is even;
    if $f(x)$ is even, $f(x)cos(nx)$ is even
  - the product of two odd functions is even;
    if $f(x)$ is odd, $f(x)sin(nx)$ is even
  - the product of an even and an odd function is odd;
    if $f(x)$ is even, $f(x)sin(nx)$ is odd
    if $f(x)$ is odd, $f(x)cos(nx)$ is odd

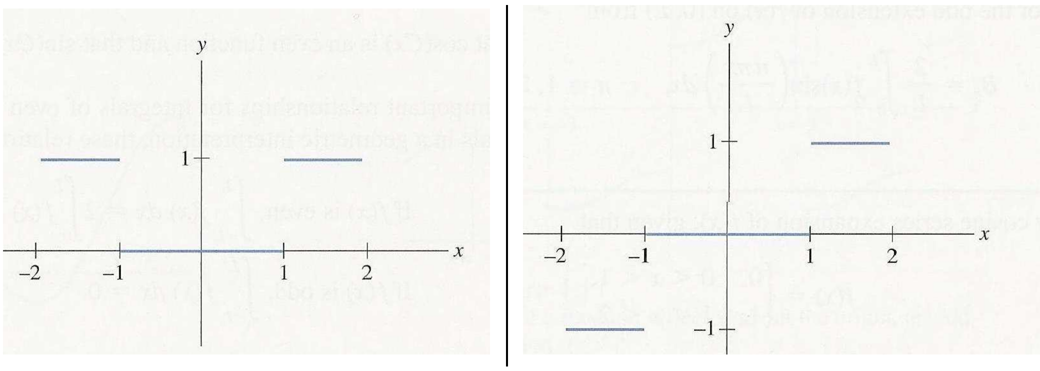

Figure 7: **Left:** Plot of the function reflected about the y-axis, **Right:** Plot of the function reflected about the origin.

- The Fourier series expansion of an even function will contain only cosine terms (all the B-coefficients are zero).
- The Fourier series expansion of an odd function will contain only sine terms (all the A-coefficients are zero).

• If we want to represent $f(x)$ between 0 and L as a Fourier series and are interested only in approximating it on the interval $(0, L)$, we can redefine f within the interval $(-L, L)$ in two importantly different ways;

  - We can redefine the portion from $-L$ to 0 byreflectingg about the y-axis. We then generate an even function.
  - We can reflect the portion between 0 and L about the origin to generate an odd function.

  Figure 7 showed these two possibilities.

• Thus two different Fourier series expansions of $f(x)$ on $(0, L)$ are possible, one that has only cosine terms or one that has only sine terms. We get the As for the even extension of $f(x)$ on $(0, L)$ from

$$A_n = \frac{2}{L}\int_0^L f(x) cos\left(\frac{n\pi x}{L}\right) dx,\ n = 0, 1, 2, \ldots$$

We get the Bs for the odd extension of $f(x)$ on $(0, L)$ from

$$B_n = \frac{2}{L}\int_0^L f(x) sin\left(\frac{n\pi x}{L}\right) dx,\ n = 1, 2, 3, \ldots$$

Examples:

1. Find the Fourier cosine series expansion of $f(x)$, given that

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$$

Figure 7left shows the even extension of the function. Because we are dealing with an even function on $(-2, 2)$ we know that the Fourier series will have only cosine terms. We get the As with

$$A_0 = \frac{2}{2}\int_1^2 (1)dx = 1$$

$$A_n = \frac{2}{2}\int_1^2 (1)cos\left(\frac{n\pi x}{2}\right) = \begin{cases} 0, & n\ even \\ \frac{2(-1)^{(n+1)/2}}{n\pi}, & n\ odd \end{cases}$$

Then the Fourier cosine series is

$$f(x) \approx \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}(\frac{(-1)^n cos\left((2n-1)\pi x/2\right)}{(2n-1)}$$

2. Find the Fourier sine series expansion for the same function. Figure 7right shows the odd extension of the function. We know that all of the A-coefficients will be zero, so we need to compute only the Bs;

$$B_n = \frac{2}{2}\int_1^2 (1)sin(\frac{n\pi x}{2})dx = \frac{2}{n\pi}\left[-cos(n\pi) + cos(\frac{n\pi}{2})\right],\ n = 1, 2, 3, \ldots$$

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[cos(n\pi/2) - cos(n\pi)]}{n}sin(\frac{n\pi x}{2})$$

1.3 Summary

- A function that is periodic of period P and meets certain criteria (see below) can be represented by Eq. 9;

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n cos\left(\frac{n\pi x}{P/2}\right) + \sum_{n=1}^{\infty} B_n sin\left(\frac{n\pi x}{P/2}\right) \tag{9}$$

  The coefficients can be computed with

$$A_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)cos\left(\frac{n\pi x}{P/2}\right)dx,\ n = 0, 1, 2, \ldots$$

$$B_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)sin\left(\frac{n\pi x}{P/2}\right)dx,\ n = 1, 2, 3, \ldots$$

  (The limits of the integrals can be from a to $a + P$)

- If $f(x)$ is an even function, only the As will be nonzero. Similarly, if $f(x)$ is odd, only the Bs will be nonzero. If $f(x)$ is neither even nor odd, its Fourier series will contain both cosine and sine terms.

- Even if $f(x)$ is not periodic, it can be represented on just the interval $(0, L)$ by redefining the function over $(-L, 0)$ by reflecting $f(x)$ about the y-axis or, alternative1y, about the origin. The first creates an even function, the second an odd function. The Fourier series of the redefined function will actually represent a periodic function of period $2L$ that is defined for $(-L, L)$.

- When L is the half-period, the Fourier series of an even function contains only cosine terms and is called a *Fourier cosine series*. The As can be computed by

$$A_n = \frac{2}{L}\int_{0}^{L} f(x)cos\left(\frac{n\pi x}{L}\right)dx,\ n = 0, 1, 2, \ldots$$

  The Fourier series of an odd function contain Ls only sine terms and is called a $Fouriersineseries$. The Bs can be computed by

$$B_n = \frac{2}{L}\int_{0}^{L} f(x)sin\left(\frac{n\pi x}{L}\right)dx,\ n = 1, 2, 3, \ldots$$