Lecture 10

Approximation of Functions I

Chebyshev Polynomials and Series

Ceng375 *Numerical Computations* at December 23, 2010

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Contents

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- We want the approximation to be efficient in that it obtains the values with the ***smallest error*** in the ***least number of arithmetic operations***.
- Another way to approximate a function is with a series of *sine* and *cosine* terms, Fourier series (represents *periodic* functions).

Approximation of Functions II

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- **Fourier Series:** These are series of sine and cosine terms that can be used to approximate a function within a given interval very closely, even functions with discontinuities.
- Fourier series are important in many areas, particularly in getting an analytical solution to partial-differential equations.

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- The error after the $(x-a)^n$ term,

$$Error = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi), where\ \xi\ in\ [a, x]$$

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$$e^x \approx 1 + 1(x - 0) + 1/2(x - 0)^2 + 1/6(x - 0)^3$$

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- There is a way to deal with this rapid growth of the errors,
- That is to write the polynomial approximation to $f(x)$ in terms of *Chebyshev polynomials*.

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- Chebyshev polynomials are not as simple; the first 11 of these are

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T_0(x) &= 1 \\
T_1(x) &= x \\
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T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \\
T_9(x) &= 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x \\
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- In Fig. 1, we plot the first four polynomials of Eqn.1.

Chebyshev Polynomials II

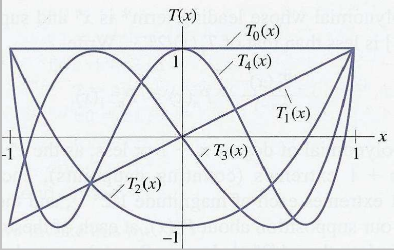

Figure: Plot of the first four polynomials of the Chebyshev polynomials.

- The members of this series of polynomials can be generated from the two-term recursion formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$
$$T_0(x) = 1 \quad \& \quad T_1(x) = x$$

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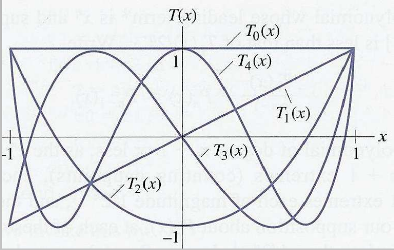

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- They form an orthogonal set,

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = \left\{ \begin{array}{ll} 0, & n \neq m \\ \pi, & n = m = 0 \\ \pi/2, & n = m \neq 0 \end{array} \right\}$$

Chebyshev Polynomials III

- The Chebyshev polynomials are also terms of a Fourier series, because

$$T_n(x) = cos(n\theta)$$

where $\theta = arc(cosx)$. Observe that

$$\begin{array}{lcr} n = 0; & cos0 = 1 \rightarrow & T_0 = 1 \\ n = 1; & cos\theta = cos(arc(cosx)) = x \rightarrow & T_1 = x \end{array}$$

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- Because of the relation $T_n(x) = cos(n\theta)$, the Chebyshev polynomials will have a succession of maxima and minima of alternating signs, as Figure 1 shows.

Chebyshev Polynomials IV

- MATLAB has no commands for these polynomials but this M-file will compute them:

```
function T=Tch(n)
if n==0
        disp('1')
elseif n==1
        disp('x')
else
        t0='1';
        t1='x';
        for i=2:n
                T=symop('2*x','*',t1,'-',t0);
                t0=t1;
                t1=T;
        end
end
>>Tch(5)
>>collect(ans)
ans= 16*x^5-20*x^3+5*x
```

if *symop* does not exist, download.

Chebyshev Polynomials V

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has a smaller upper bound to its magnitude in the interval [-1, 1].

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- **Example m-file:** Comparison of Lagrangian interpolation polynomials for equidistant and non-equidistant (Chebyshev) sample points for the function $f(x) = \frac{1}{1+x^2}$ (lagrange_chebyshev.m)

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Economizing a Power Series I

- We begin a search for better power series representations of functions by using Chebyshev polynomials to *economize* a Maclaurin series.
- This will give a modification of the Maclaurin series that produces a fifth-degree polynomial
- whose errors are only slightly greater than those of a sixth-degree Maclaurin series.

Economizing a Power Series II

- We start with a Maclaurin series for e^x:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \dots$$

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- Suppose we subtract

$$\left(\frac{1}{720}\right)\left(\frac{T_6}{32}\right)$$

from the truncated series.

Economizing a Power Series III

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- and at the same time make adjustments in other coefficients of the Maclaurin series.
- Because the maximum value of T_6 on the interval $[0, 1]$ is unity,
- this will change the sum of the truncated series by only

$$\left(\frac{1}{720}\right)\left(\frac{1}{32}\right) < 0.00005$$

which is small with respect to our required precision of 0.001.

Economizing a Power Series IV

- Performing the calculations, we have

$$e^x \approx \overbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}}^{Maclaurin}$$

$$- \left(\tfrac{1}{720}\right) \underbrace{\left(\frac{(32x^6 - 48x^4 + 18x^2 - 1)}{32}\right)}_{Chebyshev \;\; T_6(x)/32}$$

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- The resulting fifth-degree polynomial approximates e^x on $[0, 1]$ nearly as well as the sixth-degree Maclaurin series.
- its maximum error (at $x = 1$) is 0.000270, compared to 0.000226 for the Maclaurin polynomial (see Table 1).

Economizing a Power Series V

x	e^x	Maclaurin of degree 6	Maclaurin of degree 5	Maclaurin of degree 4	Economized of degree 5	Economized of degree 4
0.0	1.00000	1.00000	1.00000	1.00000	1.00004	1.00004
0.2	1.22140	1.22140	1.22140	1.22140	1.22142	1.22098
0.4	1.49182	1.49182	1.49182	1.49173	1.49178	1.49133
0.6	1.82212	1.82212	1.82205	1.82140	1.82208	1.82212
0.8	2.22554	2.22549	2.32513	2.22240	2.22553	2.22605
1.0	2.71828	2.71806	2.71667	2.70833	2.71801	2.71749
Maximum error		0.00023	0.00162	0.00995	0.00027	0.00078

Table: Comparison of economized series with Maclaurin series.

- We *economize* in that we get about the same precision with a lower-degree polynomial.

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- We *economize* in that we get about the same precision with a lower-degree polynomial.
- By subtracting $\frac{1}{120}\frac{T_5}{16}$ we can economize further, getting a fourth-degree polynomial that is almost as good as the economized fifth-degree one.

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- So that we have found a fourth-degree power series that meets an error criterion that requires us to use two additional terms of the original Maclaurin series.

Economizing a Power Series VI

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- Observe that even the economized polynomial of degree-4 is more accurate than a fifth-degree Maclaurin series.
- Also notice that near $x = 0$, the economized polynomials are less accurate.

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- We can get the economized series with MATLAB by employing our M-file for the Chebyshev series.

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- We can get the economized series with MATLAB by employing our M-file for the Chebyshev series.
- We must start with *x* as a symbolic variable, then get the Maclaurin series and subtract the proper multiple of the Chebyshev series:

```
>> syms x
>> ts=taylor(exp(x),7)
1+x+1/2*x^2+1/6*x^3+1/24*x^4+1/120*x^5+1/720*x^6
>> cs=Tch(6);
>> es=ts-cs/factorial(6)/2^5
es=23041/23040+x+639/1280*x^2+1/6*x^3+7/160*x^4+1/120*x^5
>> vpa(es,7)
>> collect(ans)
```

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x^3 &= \tfrac{1}{4}(3T_1 + T_3) \\
x^4 &= \tfrac{1}{8}(3T_0 + 4T_2 + T_4) \\
x^5 &= \tfrac{1}{16}(10T_1 + 5T_3 + T_5) \\
x^6 &= \tfrac{1}{32}(10T_0 + 15T_2 + 6T_4 + T_6) \\
x^7 &= \tfrac{1}{64}(35T_1 + 21T_3 + 7T_5 + T_7) \\
x^8 &= \tfrac{1}{128}(35T_0 + 56T_2 + 28T_4 + 8T_6 + T_8) \\
x^9 &= \tfrac{1}{256}(126T_1 + 84T_3 + 36T_5 + 9T_7 + T_9)
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- By substituting these identities into an infinite Taylor series
- and collecting terms in $T_i(x)$, we create a Chebyshev series.

Chebyshev Series II

- For example, we can get the first four terms of a Chebyshev series

$$e^x = 1.2661T_0 + 1.1302T_1 + 0.2715T_2 + 0.0443T_3$$

by starting with the Maclaurin expansion for e^x.

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- Such a series **converges more rapidly** than does a Taylor series on $[-1, 1]$;

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- The number of terms that are employed determines the accuracy of the computed values.
- To compare the Chebyshev expansion with the Maclaurin series, we convert back to powers of x, using Eqn. 1:

$$e^x = 0.9946 + 0.9973x + 0.5430x^2 + 0.1772x^3 + \ldots \quad (3)$$

Chebyshev Series III

x	e^x	Chebyshev	Error	Maclaurin	Error
−1.0	0.3679	0.3631	0.0048	0.3333	0.0346
−0.8	0.4493	0.4536	−0.0042	0.4346	0.0147
−0.6	0.5488	0.5534	−0.0046	0.5440	0.0048
−0.4	0.6703	0.6712	−0.0009	0.6693	0.0010
−0.2	0.8187	0.8154	0.0033	0.8187	0.0001
0	1.0000	0.9946	0.0054	1.0000	0.0000
0.2	1.2214	1.2172	0.0042	1.2213	0.0001
0.4	1.4918	1.4917	0.0001	1.4907	0.0012
0.6	1.8221	1.8267	−0.0046	1.8160	0.0061
0.8	2.2255	2.2307	−0.0051	2.2054	0.0202
1.0	2.7183	2.7121	0.0062	2.6667	0.0516

Table: Comparison of Chebyshev series for e^x with Maclaurin series.

- Table 2 and Figure 2 compare the error of the Chebyshev expansion ($0.9946 + 0.9973x + 0.5430x^2 + 0.1772x^3$) with the Maclaurin series ($1 + x + 0.5x^2 + 0.1667x^3$).

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  - Chebyshev expansion, the errors can be considered to be distributed more or less **uniformly throughout the interval**.
  - Maclaurin expansion, which gives very small errors near the origin, allows the error to bunch up at the ends of the interval.

Chebyshev Series IV

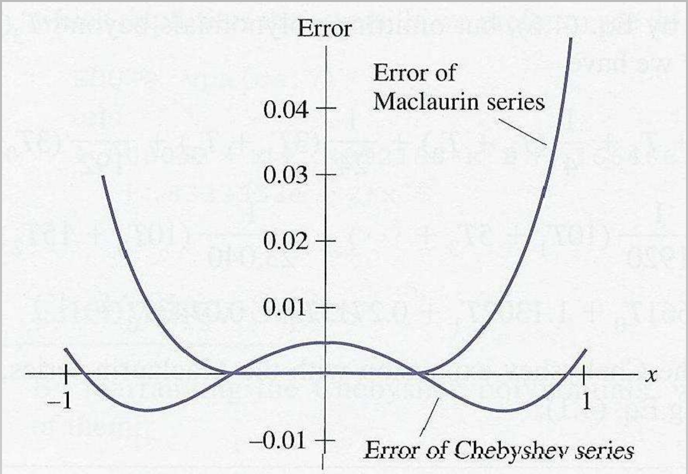

Figure: Comparison of the error of Chebyshev series for e^x with the error of Maclaurin series.