Lecture 11

Approximation of Functions II

Fourier Series

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Contents

Approximation of Functions

- A second topic, representing a function with a series of *sine* and *cosine* terms.
- A Fourier series, is usually the best way to represent a *periodic* function, something that cannot be done with a polynomial or a Taylor series.
- A Fourier series can even approximate functions with discontinuities and discontinuous derivatives.
- **Fourier Series:** These are series of sine and cosine terms that can be used to approximate a function within a given interval very closely, even functions with discontinuities. Fourier series are important in many areas, particularly in getting an analytical solution to partial-differential equations.

Fourier Series I

- Polynomials are not the only functions that can be used to approximate the known function.
- Another means for representing known functions are approximations that use *sines* and *cosines*, called **Fourier series**.
  - Any function can be represented by an *infinite* sum of sine and cosine terms with the proper coefficients, (possibly with an infinite number of terms).
- Any function, $f(x)$, is *periodic* of period P if it has the same value for any two x-values, that differ by P, or

$$f(x) = f(x+P) = f(x+2P) = \ldots = f(x-P) = f(x-2P) = \ldots$$

- Figure 1 shows such a periodic function. Observe that the period can be started at any point on the x-axis.

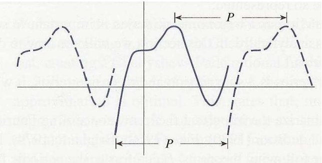

Figure: Plot of a periodic function of period P.

Fourier Series II

- $Sin(x)$ and $cos(x)$ are periodic of period 2π
- $Sin(2x)$ and $cos(2x)$ are periodic of period π
- $Sin(nx)$ and $cos(nx)$ are periodic of period $2\pi/n$
- We now discuss how to find the As and Bs in a Fourier series of the form

$$f(x) \approx \frac{A_0}{2} + \sum_{n=1}^{\infty}[A_n cos(nx) + B_n sin(nx)] \qquad (1)$$

- The determination of the coefficients of a Fourier series (when a given function, $f(x)$, can be so represented) is based on the *property of orthogonality* for sines and cosines.
- For integer values of n, m:

$$\int_{-\pi}^{\pi} sin(nx)dx = 0 \qquad (2)$$

$$\int_{-\pi}^{\pi} cos(nx)dx = \left\{ \begin{array}{ll} 0, & n \neq 0 \\ 2\pi, & n = 0 \end{array} \right\} \qquad (3)$$

Fourier Series III

$$\int_{-\pi}^{\pi} sin(nx)cos(mx)dx = 0 \tag{4}$$

$$\int_{-\pi}^{\pi} sin(nx)sin(mx)dx = \left\{ \begin{array}{ll} 0, & n \neq m \\ \pi, & n = m \end{array} \right\} \tag{5}$$

$$\int_{-\pi}^{\pi} cos(nx)cos(mx)dx = \left\{ \begin{array}{ll} 0, & n \neq m \\ \pi, & n = m \end{array} \right\} \tag{6}$$

- It is related to the same term used for orthogonal (perpendicular) vectors whose dot product is zero.
- Many functions, besides sines and cosines, are orthogonal (such as the Chebyshev polynomials).
- To begin, we assume that $f(x)$ is periodic of period 2π and can be represented as in Eq. 1.
- We find the values of A_n and B_n in Eq. 1 in the following way;
- For A_0; multiply both sides of Eq. 1 by $cos(0x) = 1$, and integrate term by term between the limits of $-\pi$ and π.

Fourier Series III

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{A_0}{2}dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} A_n cos(nx)dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} B_n sin(nx)dx$$

- Because of Eqs. 2 and 3, every term on the right vanishes except the first, giving

$$\int_{-\pi}^{\pi} f(x)dx = \frac{A_0}{2}(2\pi), \ or \ A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

  Hence, A_0 is found and it is equal to twice the average value of $f(x)$ over one period.

- For A_n; multiply both sides of Eq. 1 by $cos(mx)$, where m is any positive integer, and integrate:

$$\int_{-\pi}^{\pi} cos(mx)f(x)dx = \int_{-\pi}^{\pi} \frac{A_0}{2}cos(mx)dx +$$

$$\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} A_n cos(mx)cos(nx)dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} B_n cos(mx)sin(nx)dx$$

Fourier Series IV

- Because of Eqs. 3,4 and 6 the only nonzero term on the right is when $m = n$ in the first summation, so we get a formula for the As;

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx,\ n = 1, 2, 3, \ldots$$

- For B_n; multiply both sides of Eq. 1 by $sin(mx)$, where m is any positive integer, and integrate:

$$\int_{-\pi}^{\pi} sin(mx)f(x)dx = \int_{-\pi}^{\pi} \frac{A_0}{2}sin(mx)dx +$$

$$\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} A_n sin(mx)cos(nx)dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} B_n sin(mx)sin(nx)dx$$

- Because of Eqs. 2, 4 and 5, the only nonzero term on the right is when $m = n$ in the second summation, so we get a formula for the Bs;

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx,\ n = 1, 2, 3, \ldots$$

- It is obvious that getting the coefficients of Fourier series involves *many integrations*.

Fourier Series for Periods Other Than 2π I

- What if the period of $f(x)$ is not 2π?
- We just make a change of variable.
- If $f(x)$ is periodic of period P, the function can be considered to have one period between $-P/2$ and $P/2$.
- The functions $sin(2\pi x/P)$ and $cos(2\pi x/P)$ are periodic between $-P/2$ and $P/2$.
- The formulae become, for $f(x)$ periodic of period P;

$$A_n = \frac{1}{P/2}\int_{-P/2}^{P/2} f(x) cos\left(\frac{\pi}{P/2}nx\right) dx,\ n = 0, 1, 2, \ldots \quad (7)$$

$$B_n = \frac{1}{P/2}\int_{-P/2}^{P/2} f(x) sin\left(\frac{\pi}{P/2}nx\right) dx,\ n = 1, 2, 3, \ldots \quad (8)$$

- Because a function that is periodic with period P between $-P/2$ and $P/2$ is also periodic with period P between A and $A+P$, the limits of integration in Eqs. 7 and 8 can be from 0 to P.

Fourier Series for Periods Other Than 2π II

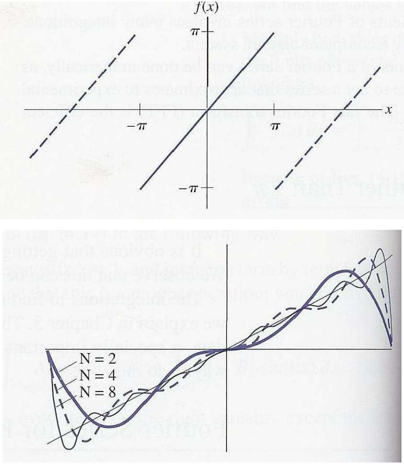

Figure: Upper: Plot of $f(x) = x$, periodic of period 2π,**Lower:** Plot of the Fourier series expansion for $N = 2, 4, 8$.

Fourier Series for Periods Other Than 2π III

Examples:

1. Let $f(x) = x$ be periodic between $-\pi$ and π. (See Figure 2upper). Find the As and Bs of its Fourier expansion.
   - For A_0;

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} xdx = \left[\frac{x^2}{2\pi}\right]_{-\pi}^{\pi} = 0$$

   - For the other As;

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} xcos(nx)dx = 0$$

   - For the other Bs;

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} xsin(nx)dx = \frac{2(-1)^{n+1}}{n},\ n = 1, 2, 3, \ldots$$

   - We then have

$$x \approx 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} sin(nx),\ -\pi < x < \pi$$

   Figure 2lower shows how the series approximates to the function when only two, four, or eight terms are used.

Fourier Series for Periods Other Than 2π IV

2 Find the Fourier coefficients for $f(x) = |x|$ on $-\pi$ to π;

- $$A_0 = \frac{1}{\pi}\int_{-\pi}^{0} -x dx + \frac{1}{\pi}\int_{0}^{\pi} x dx = \pi$$

- $$A_n = \frac{1}{\pi}\int_{-\pi}^{0} (-x) cos(nx) dx + \frac{1}{\pi}\int_{0}^{\pi} (x) cos(nx) dx =$$
$$\left\{ \begin{array}{ll} 0, & n = 2, 4, 6, \ldots \\ \frac{-4}{(n^2\pi)}, & n = 1, 3, 5, \ldots \end{array} \right\}$$

- $$B_n = \frac{1}{\pi}\int_{-\pi}^{0} (-x) sin(nx) dx + \frac{1}{\pi}\int_{0}^{\pi} (x) sin(nx) dx = 0$$

  Because the definite integrals are nonzero only for odd values of n, it simplifies to change the index of the summation. The Fourier series is then

$$|x| \approx \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{cos(2n-1)x}{(2n-1)^2}$$

Fourier Series for Periods Other Than 2π V

Figure 3 shows how the series approximates the function when two, four, or eight terms are used.

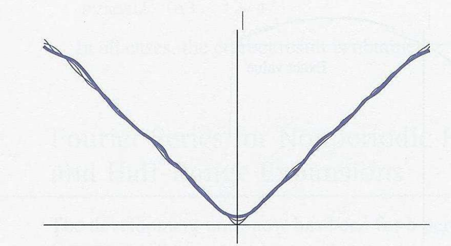

Figure: Plot of Fourier series for $|x|$ for $N = 2, 4, 8$.

Fourier Series for Periods Other Than 2π VI

3 Find the Fourier coefficients for $f(x) = x(2-x) = 2x - x^2$ over the interval [-2, 2] if it is periodic of period 4. Equations 7 and 8 apply.

- $$A_0 = \frac{2}{4}\int_{-2}^{2}(2x - x^2)dx = \frac{-8}{3}$$
- $$A_n = \frac{2}{4}\int_{-2}^{2}(2x - x^2)cos\left(\frac{n\pi x}{2}\right)dx = \frac{16(-1)^{n+1}}{n^2\pi^2},\ n = 1, 2, 3, \ldots$$
- $$B_n = \frac{2}{4}\int_{-2}^{2}(2x - x^2)sin\left(\frac{n\pi x}{2}\right)dx = \frac{8(-1)^{n+1}}{n\pi},\ n = 1, 2, 3, \ldots$$

$$x(2-x) \approx \frac{-4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}cos\left(\frac{n\pi x}{2}\right) +$$

$$\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}sin\left(\frac{n\pi x}{2}\right)$$

Fourier Series for Periods Other Than 2π VII

- Figure 4 shows how the series approximates to the function when 40 terms are used.

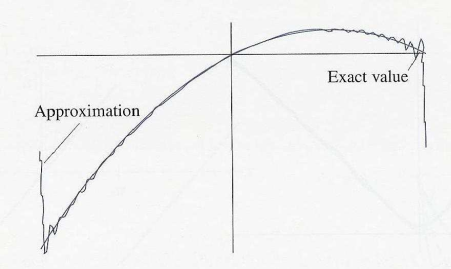

Figure: Plot of Fourier series for $x(2-x)$ for $N = 40$.

- With MATLAB,

```
>>a3=int('2/4*x*(2-x)*cos(3*pi*x/2)',-2,2)
a3 = 16/9/pi^2
>>b3=int('2/4*x*(2-x)*sin(3*pi*x/2)',-2,2)
b3 = 8/3/pi
```

Fourier Series for Nonperiodic Functions and Half-Range Expansions I

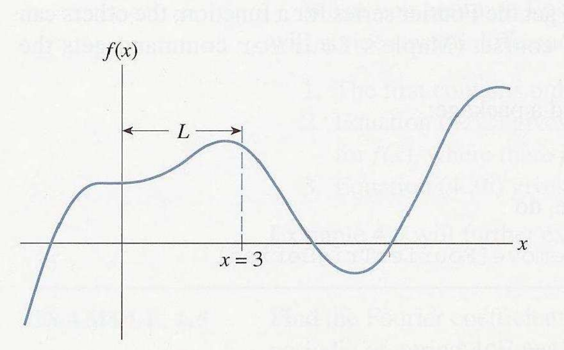

Figure: A function, $f(x)$, of interest on [0,3].

Fourier Series for Nonperiodic Functions and Half-Range Expansions II

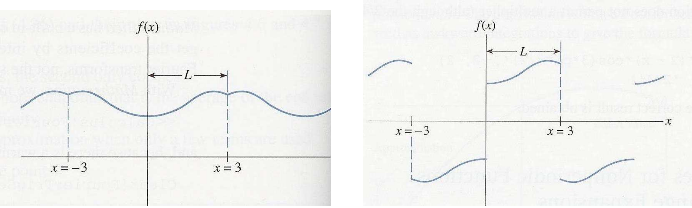

Figure: Left: Plot of a function reflected about the y-axis, an even function,**Right:** Plot of a function reflected about the origin, an odd function.

- The development until now has been for a periodic function. What if $f(x)$ is *not* periodic?
- Can we approximate it by a trigonometric series?
- We assume that we are interested in approximating the function only over a limited interval
- and we do not care whether the approximation holds outside of that interval.

Fourier Series for Nonperiodic Functions and Half-Range Expansions III

- Suppose we have a function defined for all x-values, but we are only interested in representing it over (0, L). Figure 5 is typical.
- Because we will ignore the behavior of the function outside of (0, L),
- we can redefine the behavior outside that interval as we wish Figs. 6left and -right show two possible redefinitions.
  - In the first redefinition, we have reflected the portion of $f(x)$ about the y-axis and have extended it as a periodic function of period $2L$. This creates an *even* periodic function.

$$f(x) \text{ is even if } f(-x) = f(x)$$

  - If we reflect it about the origin and extend it periodically, we create an odd periodic function of period $2L$.

$$f(x) \text{ is odd if } f(-x) = -f(x)$$

- It is easy to see that $cos(Cx)$ is an even function and that $sin(Cx)$ is an odd function for any real value of C.

Fourier Series for Nonperiodic Functions and Half-Range Expansions IV

- There are two important relationships for integrals of even and odd functions.

$$\begin{array}{ll} \text{if } f(x) \text{ is even}, & \int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx \\ \text{if } f(x) \text{ is odd}, & \int_{-L}^{L} f(x)dx = 0 \end{array}$$

- the product of two even functions is even;
  if $f(x)$ is even, $f(x)cos(nx)$ is even
- the product of two odd functions is even;
  if $f(x)$ is odd, $f(x)sin(nx)$ is even
- the product of an even and an odd function is odd;
  if $f(x)$ is even, $f(x)sin(nx)$ is odd
  if $f(x)$ is odd, $f(x)cos(nx)$ is odd
- The Fourier series expansion of an <u>even function</u> will contain <u>only cosine terms</u> (all the *B*-coefficients are zero).
- The Fourier series expansion of an <u>odd</u> function will contain <u>only sine terms</u> (all the *A*-coefficients are zero).

Fourier Series for Nonperiodic Functions and Half-Range Expansions V

- If we want to represent $f(x)$ between 0 and L as a Fourier series and are interested only in approximating it on the interval $(0, L)$,
- we can redefine f within the interval $(-L, L)$ in two importantly different ways;
  1. We can redefine the portion from $-L$ to 0 by reflecting about the y-axis. We then generate an even function.
  2. We can reflect the portion between 0 and L about the origin to generate an odd function.
- Figure 7 shows these two possibilities.

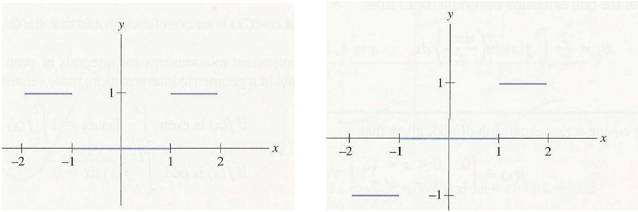

Figure: **Left:** Plot of the function reflected about the y-axis, **Right:** Plot of the function reflected about the origin.

Fourier Series for Nonperiodic Functions and Half-Range Expansions VI

- Thus two different Fourier series expansions of $f(x)$ on $(0, L)$ are possible,
  1. one that has only cosine terms
  2. one that has only sine terms.
- We get the As for the even extension of $f(x)$ on $(0, L)$ from

$$A_n = \frac{2}{L}\int_0^L f(x)cos\left(\frac{n\pi x}{L}\right)dx,\ n = 0, 1, 2, \ldots$$

- We get the Bs for the odd extension of $f(x)$ on $(0, L)$ from

$$B_n = \frac{2}{L}\int_0^L f(x)sin\left(\frac{n\pi x}{L}\right)dx,\ n = 1, 2, 3, \ldots$$

Fourier Series for Nonperiodic Functions and Half-Range Expansions VII

Examples:

1. Find the *Fourier cosine series expansion* of $f(x)$, given that

$$f(x) = \left\{ \begin{array}{ll} 0, & 0 < x < 1 \\ 1, & 1 < x < 2 \end{array} \right\}$$

- Figure 7left shows the even extension of the function.
- Because we are dealing with an even function on $(-2, 2)$ we know that the Fourier series will have only cosine terms.
- We get the As with

$$A_0 = \frac{2}{2}\int_1^2 (1)dx = 1$$

$$A_n = \frac{2}{2}\int_1^2 (1)cos\left(\frac{n\pi x}{2}\right) = \left\{ \begin{array}{ll} 0, & n\ even \\ \frac{2(-1)^{(n+1)/2}}{n\pi}, & n\ odd \end{array} \right\}$$

- Then the Fourier cosine series is

$$f(x) \approx \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}(\frac{(-1)^n cos\left((2n-1)\pi x/2\right)}{(2n-1)}$$

Fourier Series for Nonperiodic Functions and Half-Range Expansions VIII

2 Find the *Fourier sine series expansion* for the same function. Figure 7right shows the odd extension of the function.

- We know that all of the A-coefficients will be zero, so we need to compute only the Bs;

$$B_n = \frac{2}{2}\int_1^2 (1) sin(\frac{n\pi x}{2}) dx = \frac{2}{n\pi}\left[-cos(n\pi) + cos(\frac{n\pi}{2})\right]$$

$$n = 1, 2, 3, \ldots$$

-

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[cos(n\pi/2) - cos(n\pi)]}{n} sin(\frac{n\pi x}{2})$$

Summary I

- A function that is periodic of period P and meets certain criteria (see below) can be represented by Eq. 9;

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n cos\left(\frac{n\pi x}{P/2}\right) + \sum_{n=1}^{\infty} B_n sin\left(\frac{n\pi x}{P/2}\right) \qquad (9)$$

  The coefficients can be computed with

$$A_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x) cos\left(\frac{n\pi x}{P/2}\right) dx, \; n = 0, 1, 2, \ldots$$

$$B_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x) sin\left(\frac{n\pi x}{P/2}\right) dx, \; n = 1, 2, 3, \ldots$$

  (The limits of the integrals can be from a to $a + P$)
- If $f(x)$ is an even function, only the As will be nonzero.
- Similarly, if $f(x)$ is odd, only the Bs will be nonzero.
- If $f(x)$ is neither even nor odd, its Fourier series will contain **both** cosine and sine terms.

Summary II

- Even if $f(x)$ is not periodic, it can be represented on just the interval $(0, L)$ by redefining the function over $(-L, 0)$ by reflecting $f(x)$ about the y-axis or, alternative1y, about the origin.
- The first creates an even function, the second an odd function.
- The Fourier series of the redefined function will actually represent a periodic function of period $2L$ that is defined for $(-L, L)$.
- When L is the half-period, the Fourier series of an even function contains only cosine terms and is called a *Fourier cosine series*. The As can be computed by

$$A_n = \frac{2}{L}\int_0^L f(x) cos\left(\frac{n\pi x}{L}\right) dx,\ n = 0, 1, 2, \ldots$$

The Fourier series of an odd function contain Ls only sine terms and is called a *Fourier sine series*. The Bs can be computed by

$$B_n = \frac{2}{L}\int_0^L f(x) sin\left(\frac{n\pi x}{L}\right) dx,\ n = 1, 2, 3, \ldots$$