Lecture 4

Solving Nonlinear Equations II

Roots of the equation, Convergence

Ceng375 *Numerical Computations* at October 21, 2010

Dr. Cem Özdoğan
Computer Engineering Department
Çankaya University

Contents

Solving Nonlinear Equations
Newton's Method, Continued
Muller's Method
Fixed-point Iteration; $x = g(x)$ Method
Other Rearrangements
Order of Convergence
Multiple Roots
The fzero function
Nonlinear Systems
Solving a System by Iteration

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- Multiple Roots. Nonlinear Systems

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- When Newton's method is applied to $f(x) = 3x + sinx - e^x = 0$, if we begin with $x_0 = 0.0$:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.0 - \frac{-1.0}{3.0} = 0.33333$$

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- In fact, the error after an iteration is about one-third of the square of the previous error.

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- Because Newton's method always uses two per iteration whereas the others take only one.
- **An algorithm for the Newton's method :**

To determine a root of $f(x) = 0$, given x_0 reasonably close to the root,
Compute $f(x_0), f^{'}(x_0)$
If $(f(x_0) \neq 0)$ And $(f^{'}(x_0) \neq 0)$ Then
Repeat
Set $x_1 = x_0$
Set $x_0 = x_0 - \frac{f(x_0)}{f^{'}(x_0)}$
Until $(|x_1 - x_0| < tolerance\ value1)$ Or If $|f(x_0)| < tolerance\ value2)$
End If.

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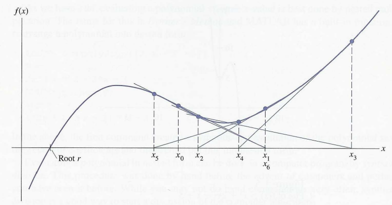

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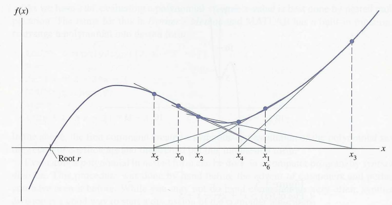

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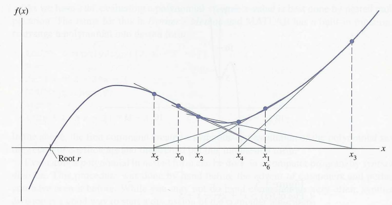

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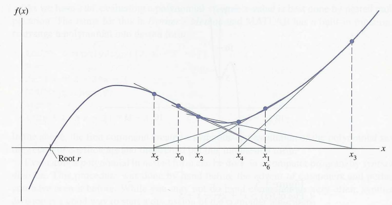

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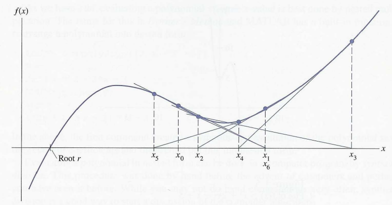

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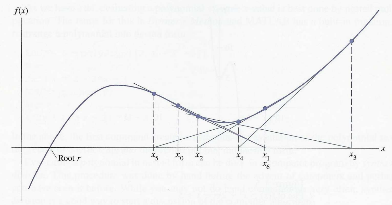

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- **Example**: A general implementation of Newton's method. (**m-files: newton.m**),(**fx3n.m**). » newton('fx3n',3,5e-16,5e-16,1)

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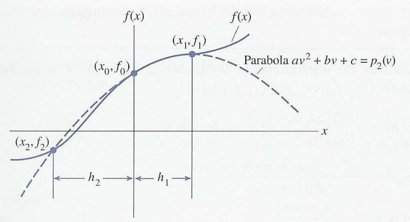

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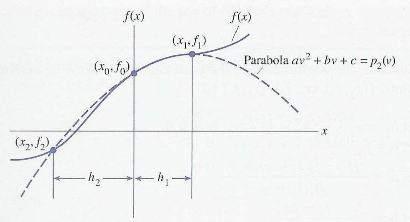

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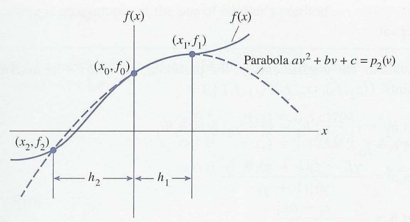

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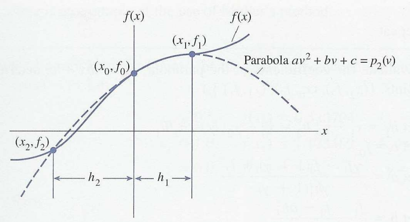

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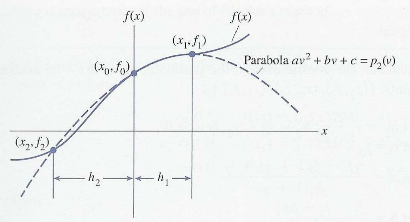

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- A quadratic equation that fits through three points in the vicinity of a root, in the form $a\nu^2 + b\nu + c$. (See Fig. 2)

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$$\nu = -h_2 \implies ah_2^2 - bh_2 + c = f_2$$

We evaluate the coefficients by evaluating $p_2(\nu)$ at the three points:

- From the first equation, $c = f_0$.
- Letting $h_2/h_1 = \gamma$, we can solve the other two equations for a, and b.

$$a = \frac{\gamma f_1 - f_0(1+\gamma) + f_2}{\gamma h_1^2(1+\gamma)}, \; b = \frac{f_1 - f_0 - ah_1^2}{h_1}$$

- After computing a, b, and c, we solve for the root of $a\nu^2 + b\nu + c$ by the quadratic formula

$$\nu_{1,2} = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}, \; \nu = x - x_0, \quad root = x_0 - \frac{2c}{b \pm \sqrt{b^2 - 4ac}}$$

Muller's Method III

See Figs. 3-4 that an example is given

Find a root between 0 and 1 of the same transcendental function as before: $f(x) = 3x + \sin(x) - e^x$. Let

$$x_0 = 0.5, \quad f(x_0) = 0.330704 \quad h_1 = 0.5,$$
$$x_1 = 1.0, \quad f(x_1) = 1.123489 \quad h_2 = 0.5,$$
$$x = 0.0, \quad f(x_2) = -1 \quad \gamma = 1.0.$$

Then

$$a = \frac{(1.0)(1.123189) - 0.330704(2.0) + (-1)}{1.0(0.5)^2(2.0)} = -1.07644,$$

$$b = \frac{1.123189 - 0.330704 - (-1.07644)(0.5)^2}{0.5} = 2.12319,$$

$$c = 0.330704,$$

and

$$\text{root} = 0.5 - \frac{2(0.330704)}{2.12319 + \sqrt{(2.12319)^2 - 4(-1.07644)(0.330704)}}.$$
$$= 0.354914.$$

For the next iteration, we have

$$x_0 = 0.354914, \quad f(x_0) = -0.0138066 \quad h_1 = 0.145086,$$
$$x_1 = 0.5, \quad f(x_1) = 0.330704 \quad h_2 = 0.354914,$$
$$x_2 = 0, \quad f(x_2) = -1 \quad \gamma = 2.44623.$$

Figure: An example of the use of Muller's method.

Muller's Method IV

Then

$$a = \frac{(2.44623)(0.330704) - (-0.0138066)(3.44623) + (-1)}{2.44623(0.145086)^2(3.44623)} = -0.808314,$$

$$b = \frac{0.330704 - (-0.0138066) - (-0.808314)(0.145086)^2}{0.145086} = 2.49180,$$

$$c = -0.0138066,$$

$$\text{root} = 0.354914 - \frac{2(-0.0138066)}{2.49180 + \sqrt{(2.49180)^2 - 4(-0.808314)(-0.0138066)}}$$

$$= 0.360465.$$

After a third iteration, we get 0.3604217 as the value for the root, which is identical to that from Newton's method after three iterations.

Figure: Cont. An example of the use of Muller's method.

- *Experience shows that Muller's method converges at a rate that is similar to that for Newton's method.*

Muller's Method IV

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$$a = \frac{(2.44623)(0.330704) - (-0.0138066)(3.44623) + (-1)}{2.44623(0.145086)^2(3.44623)} = -0.808314,$$

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After a third iteration, we get 0.3604217 as the value for the root, which is identical to that from Newton's method after three iterations.

Figure: Cont. An example of the use of Muller's method.

- *Experience shows that Muller's method converges at a rate that is similar to that for Newton's method.*
- It does not require the evaluation of derivatives, however, and (after we have obtained the starting values) needs only one function evaluation per iteration.

Muller's Method V

An algorithm for Muller's method :

Given the points x_2, x_0, x_1 in increasing value,
Evaluate the corresponding function values: f_2, f_0, f_1.
Repeat
(Evaluate the coefficients of the parabola, $a\nu^2 + b\nu + c$, determined by the three points.
$(x_2, f_2), (x_0, f_0), (x_l, f_1)$.)
Set $h_l = x_1 - x_0; h_2 = x_0 - x_2; \gamma = h_2/h_1$.
Set $c = f_0$
Set $a = \frac{\gamma f_1 - f_0(1+\gamma) + f_2}{\gamma h_1^2 (1+\gamma)}$
Set $b = \frac{f_1 - f_0 - a h_1^2}{h_1}$
(Next, compute the roots of the polynomial.)
Set $root = x_0 - \frac{2c}{b \pm \sqrt{b^2 - 4ac}}$
Choose root, x_r, closest to x_0 by making the denominator as large as possible; i.e. if
$b > 0$, choose plus; otherwise, choose minus.
If $x_r > x_0$,
Then rearrange to:x_0, x_1, and the root
Else rearrange to: x_0, x_2, and the root
End If.
(In either case, reset subscripts so that x_0, is in the middle.)
Until $|f(x_r)| < Ftol$

Fixed-point Iteration; $x = g(x)$ Method I

- Rearrange $f(x)$ into an equivalent form $x = g(x)$,

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- The iterative form:

$$x_{n+1} = g(x_n); \quad n = 0, 1, 2, 3, \ldots$$

converges to the fixed point r, a root of $f(x)$.

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- Suppose we rearrange to give this equivalent form:

$$x = g_1(x) = \sqrt{2x+3} \quad \begin{array}{lcl} x_0 = 4 & \rightarrow & x_1 = \sqrt{11} = 3.31662 \\ x_2 = \sqrt{9.63325} = 3.10375 & \rightarrow & x_3 = 3.03439 \\ x_4 = 3.01144 & \rightarrow & \underline{\underline{x_5 = 3.00381}} \end{array}$$

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- If we start with $x = 4$ and iterate with the fixed-point algorithm,
- The values are *converging on the root* at $x = 3$.

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- Another rearrangement of $f(x)$; Let us start the iterations again with $x_0 = 4$. Successive values then are:

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$$x = g_2(x) = \frac{3}{(x-2)} \quad \begin{array}{lclcl} x_0 = 4 & \rightarrow & x_1 = 1.5 & \rightarrow \\ x_2 = -6 & \rightarrow & x_3 = -0.375 & \rightarrow \\ x_4 = -1.263158 & \rightarrow & x_5 = -0.919355 & \rightarrow \\ x_5 = -0.919355 & \rightarrow & x_6 = -1.02762 & \rightarrow \\ x_7 = -0.990876 & \rightarrow & \underline{\underline{x_8 = -1.00305}} & \end{array}$$

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- Consider a third rearrangement; starting again with $x_0 = 4$, we get

$$x = g_3(x) = \frac{(x^2 - 3)}{2}$$

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- It seems that we now <u>converge to the other root</u>, at $x = -1$.
- Consider a third rearrangement; starting again with $x_0 = 4$, we get

$$x = g_3(x) = \frac{(x^2-3)}{2} \quad \begin{array}{llll} x_0 = 4 & \rightarrow & x_1 = 6.5 & \rightarrow \\ x_2 = 19.625 & \rightarrow & \underline{\underline{x_3 = 191.070}} & \end{array}$$

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- The iterations are obviously diverging.

Fixed-point Iteration; $x = g(x)$ Method III: Other Rearrangements

- The fixed point of $x = g(x)$ is the intersection of the line $y = x$ and the curve $y = g(x)$ plotted against x.

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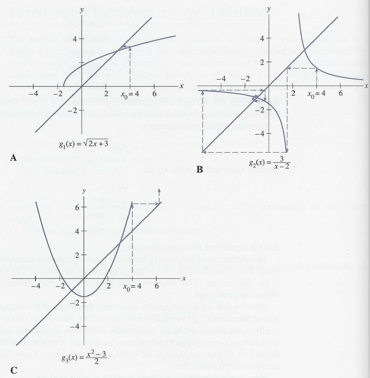

Figure: The fixed point of $x = g(x)$ is the intersection of the line $y = x$ and the curve $y = g(x)$ plotted against x. Where A:$x = g_1(x) = \sqrt{2x+3}$. B:$x = g_2(x) = \frac{3}{(x-2)}$. C: $x = g_3(x) = \frac{(x^2-3)}{2}$.

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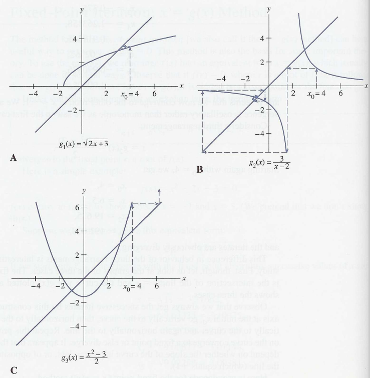

Figure 5 shows the three cases.

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Fixed-point Iteration; $x = g(x)$ Method IV: Other Rearrangements

- Start on the x-axis at the initial x_0, go vertically to the curve, then horizontally to the line $y = x$, then vertically to the curve, and again horizontally to the line.

Fixed-point Iteration; $x = g(x)$ Method IV: Other Rearrangements

- Start on the x-axis at the initial x_0, go vertically to the curve, then horizontally to the line $y = x$, then vertically to the curve, and again horizontally to the line.
- Repeat this process until the points on the curve converge to a fixed point or else diverge.

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- *The method may converge to a root different from the expected one, or it may diverge.*

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- *The method may converge to a root different from the expected one, or it may diverge.*
- *Different rearrangements will converge at different rates.*
- **Iteration algorithm with the form $x = g(x)$**

To determine a root of $f(x) = 0$, given a value x_1 reasonably close to the root
Rearrange the equation to an equivalent form $x = g(x)$
Repeat
Set $x_2 = x_l$
Set $x_l = g(x_1)$
Until $|x_1 - x_2| <$ *tolerance value*

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Table: The order of convergence for the iteration algorithm with the different forms of $x = g(x)$.

	If $g(x) = \sqrt{2x + 3}$		If $g(x) = 3/(x - 2)$	
Iteration	Error	Ratio	Error	Ratio
1	0.31662	0.31662	2.50000	0.50000
2	0.10375	0.32767	−5.00000	−2.00000
3	0.03439	0.33143	0.62500	−0.12500
4	0.01144	0.33270	−0.26316	−0.42105
5	0.00381	0.33312	0.08065	−0.30645
6			−0.02762	−0.34254
7			0.00912	−0.33029
8			−0.00305	−0.33435

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Order of Convergence

- The fixed-point method converges at a linear rate;
- it is said to be *linearly convergent*, meaning that the error at each successive iteration is a constant fraction of the previous error.

Table: The order of convergence for the iteration algorithm with the different forms of $x = g(x)$.

	If $g(x) = \sqrt{2x+3}$		If $g(x) = 3/(x-2)$	
Iteration	**Error**	**Ratio**	**Error**	**Ratio**
1	0.31662	0.31662	2.50000	0.50000
2	0.10375	0.32767	−5.00000	−2.00000
3	0.03439	0.33143	0.62500	−0.12500
4	0.01144	0.33270	−0.26316	−0.42105
5	0.00381	0.33312	0.08065	−0.30645
6			−0.02762	−0.34254
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- **Example**: Comparing Muller's and Fixed-point Iteration methods (**m-files: mainmulfix.m**, **muller.m**, **fixedpoint.m**)

Multiple Roots I

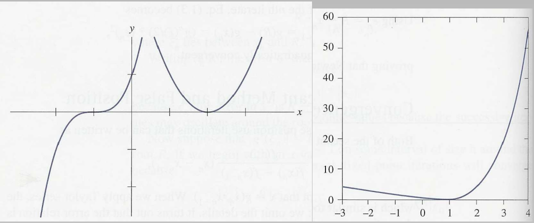

Figure: Left: The curve on the left has a triple root at $x = -1$ [the function is $(x+1)^3$]. The curve on the right has a double root at $x = 2$ [the function is $(x-2)^2$].Right: Plot of $(x-1)(e^{(x-1)}-1)$.

- A function can have more than one root of the same value. See Fig. 6left.

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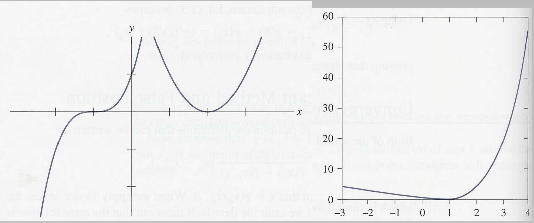

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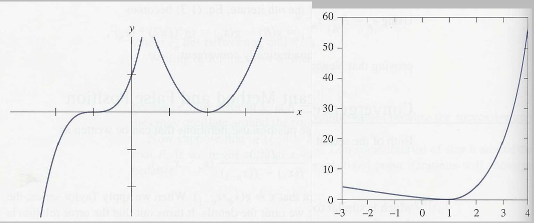

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- The methods we have described do not work well for multiple roots.

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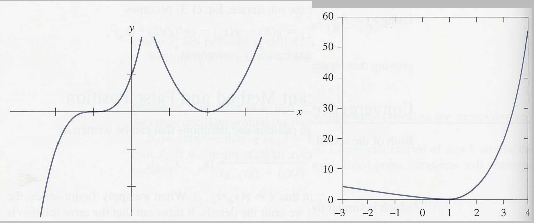

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- A function can have more than one root of the same value. See Fig. 6left.
- $f(x) = (x - 1)(e^{(x-1)} - 1)$ has a double root at $x = 1$, as seen in Fig. 6right.
- The methods we have described do not work well for multiple roots.
- For example, Newton's method is only linearly convergent at a double root.

Multiple Roots II

Table: Left: Errors when finding a double root. Right: Successive errors with Newton's method for $f(x) = (x+1)^3 = 0$ (Triple root).

Iteration	Error	Ratio
1	0.3679	
2	0.1666	0.453
3	0.0798	0.479
4	0.0391	0.490
5	0.0193	0.494
6	0.0096	0.497
7	0.0048	0.500
8	0.0024	0.500

Iteration	Error	Iteration	Error
0	0.5	6	0.0439
1	0.3333	7	0.0293
2	0.2222	8	0.0195
3	0.1482	9	0.0130
4	0.0988	10	0.00867
5	0.0658		

- Table 2left gives the errors of successive iterates (Newton's method is applied to a double root) and the convergence is clearly linear with ratio of errors is $\frac{1}{2}$.

```
>> x = linspace( -4, 4, 100 );plot(x,x.^3+3*x.^2+3*x+1); grid on
>> x= linspace( -4, 4, 100 );plot(x,x.*exp(x-1)-x-exp(x-1)+1); grid on
>> x = linspace( 0, 4, 1500 );plot(x,x.^2-4*x+4); grid on
```

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- Table 2left gives the errors of successive iterates (Newton's method is applied to a double root) and the convergence is clearly linear with ratio of errors is $\frac{1}{2}$.
- When Newton's method is applied to a triple root, convergence is still linear, as seen in Table 2right. The ratio of errors is larger, about $\frac{2}{3}$.

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The *fzero* function

- The **MATLAB** *fzero* function is a hybrid of bisection, the secant method, and interpolation.

```
>> xb=brackPlot('fx3',0,5);
>> fzero('fx3',xb)
ans =     3.5214
options=optimset('Display','iter');
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- Care is taken to *avoid unnecessary calculations and to minimize the effects of roundoff*.

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Nonlinear Systems I

Nonlinear Systems I

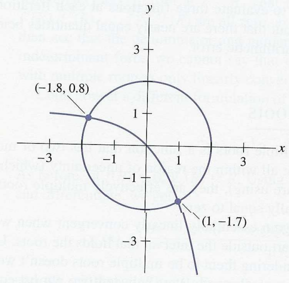
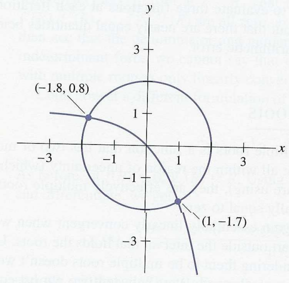


Figure: A pair of equations.

Nonlinear Systems I



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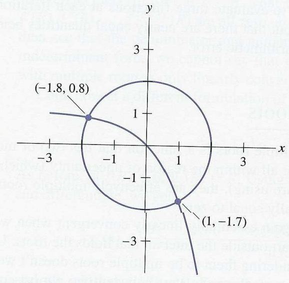

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- A pair of equations:

$$x^2 + y^2 = 4$$
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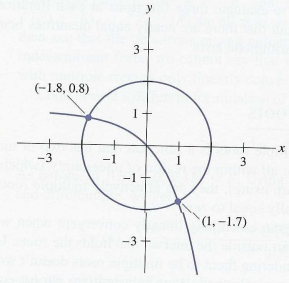

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- A pair of equations:
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- Graphically, the <u>solution</u> to this system is represented by the <u>intersections of the circle</u> $x^2 + y^2 = 4$ <u>with the curve</u> $y = 1 - e^x$ (see Fig. 7)

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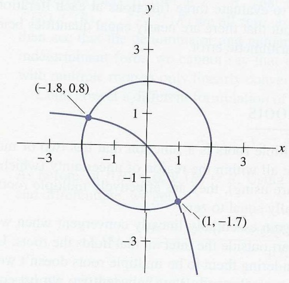

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$$f(x, y) = 0,$$
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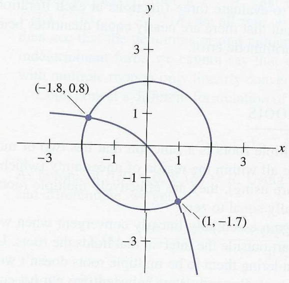

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$$f(x, y) = 0,$$
$$g(x, y) = 0$$

- Let

$$x = r, y = s$$

be a **root.**

Nonlinear Systems II

- Expand both functions as a Taylor series about the point (x_i, y_i) in terms of

$$(r - x_i), (s - y_i)$$

where (x_i, y_i) is a point near the root:

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- which we can rewrite as

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- where Δx_i and Δy_i are used as increments to x_i and y_i;

$$x_{i+1} = x_i + \Delta x_i$$

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  are improved estimates of the (x, y) values.

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- We repeat this until both $f(x, y)$ and $g(x, y)$ are close to zero.

Nonlinear Systems III

Example:

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$$
\begin{aligned}
f(x,y) &= 4 - x^2 - y^2 = 0 \\
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\end{aligned}
$$

Nonlinear Systems III

Example:

$$f(x,y) = 4 - x^2 - y^2 = 0$$
$$g(x,y) = 1 - e^x - y = 0$$

The partial derivatives are

$$f_x = -2x, f_y = -2y,$$

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- Beginning with $x_0 = 1, y_0 = -1.7$, we solve

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$$f(x,y) = 4 - x^2 - y^2 = 0$$
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The partial derivatives are

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Then,

f(1.004169,-1.729637)=-0.0000001,

g(1.004169,-1.729637)=-0.00000001,

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and begin with $x = 1.0$, the successive values for y and x are: (See Table 4)

Table: Another example for solving a system by iteration

y-value	x-value
-1.7291	1.0051
-1.72975	1.00398
-1.72961	1.00421
-1.72964	1.00416
-1.72963	1.00417

- *We are converging to the solution in an oscillatory manner.*